Duration: 2 ½ Hrs Marks: 75

N.B. : (1) All questions are compulsory.
(2) Figures to the right indicate marks.

1. (a) Attempt any One of the following: (8)

(i) Prove that the elimination of arbitrary function ϕ from the equation $\phi(u, v) = 0$, where u and v are functions of x, y and z (z is assumed to be a function of x and y), gives the partial differential equation

$$\frac{\partial(u, v)}{\partial(y, z)} * p + \frac{\partial(u, v)}{\partial(x, z)} * q = \frac{\partial(u, v)}{\partial(x, y)}.$$

Solution: Given

$$\phi(u, v) = 0 \tag{1}$$

We treat z as a function of x and y. (x and y as independent variables and z as dependent variable).
So $\dfrac{\partial y}{\partial x} = 0$ and $\dfrac{\partial x}{\partial y} = 0$
Let $\dfrac{\partial z}{\partial x} = p$ and $\dfrac{\partial z}{\partial y} = q$.
Since ϕ is a function of (u, v) and u, v are functions of x, y and z where z is a function of x and y, differentiating $\phi(u, v) = 0$ partially with respect to x, we get

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] = 0$$

$$0 = \phi_u\Big(u_x + 0 + u_z p\Big) + \phi_v\Big(v_x + 0 + v_z p\Big) = 0$$

$$\phi_u(u_x + u_z p) = -\phi_v(v_x + v_z p)$$

$$\frac{\phi_u}{\phi_v} = -\frac{v_x + v_z p}{u_x + u_z p} \tag{2}$$

Similarly, differentiating 1 with resepct to y, we get

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right] = 0$$

$$0 = \phi_u\Big(0 + u_y * 1 + u_z q\Big) + \phi_v\Big(0 + v_y * 1 + v_z q\Big) = 0$$

$$\phi_u(u_y + u_z q) = -\phi_v(v_y + v_z q)$$

$$\frac{\phi_u}{\phi_v} = -\frac{v_y + v_z q}{u_y + u_z q} \tag{3}$$

Taking ratio of (2) and (3),

$$-\frac{v_x + v_z p}{u_x + u_z p} = -\frac{v_y + v_z q}{u_y + u_z q}$$

This implies

$$(v_x + v_z p)(u_y + u_z q) = (v_y + v_z q)(u_x + u_z p)$$

$$(u_y v_z - v_y u_z)p + (u_z v_x - u_x v_z)q + u_z v_z pq = u_x v_y - u_y v_x + u_z v_z qp$$

$$(u_y v_z - v_y u_z)p + (u_z v_x - u_x v_z)q = u_x v_y - u_y v_x$$

$$\det\begin{pmatrix} u_y & u_z \\ v_y & v_z \end{pmatrix} p + \det\begin{pmatrix} u_z & u_x \\ v_z & v_x \end{pmatrix} q = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

$$\frac{\partial(u,v)}{\partial(y,z)} * p + \frac{\partial(u,v)}{\partial(z,x)} * q = \frac{\partial(u,v)}{\partial(x,y)} \tag{4}$$

(ii) If $z = F(x, y; a)$ is a one-parameter family of solutions of the partial differential equation $f(x, y, z, p, q) = 0$ where $p = z_x = F_x, q = z_y = F_y$, prove that the envelope of this family, if it exists, is also a solution of $f(x, y, z, p, q) = 0$.

Solution: (Note: This is Lemma 1.3.1 from our syllabus.)
$z = F(x, y; a)$ is a one-parameter family of solutions of $f(x, y, z, p, q) = 0$.
The envelope E of this family is obtained by eliminating a between

$$z = F(x, y; a), \tag{5}$$

$$0 = F_a(x, y; a). \tag{6}$$

The equation of E is found by solving $F_a(x, y, a) = 0$ for a as a function of x and $y, a = a(x, y)$ and then substituting into $z = f(x, y; a)$.

Hence the envelope E will be of the form $z = G(x, y) = F(x, y, a(x, y))$.
We want to show that $z = F(x, y, a(x, y))$ is a solution of $f(x, y, z, p, q) = 0$.
It is sufficient to show that $p = G_x, q = G_y$.
Now, $G(x, y) = F(x, y, a(x, y)) \implies G_x = F_x + F_a * a_x = F_x + 0 * a_x = F_x = p$ and $G_y = F_y + F_a * a_y = F_x + 0 * a_y = F_y = q$ (as $F_a = 0, F_b = 0$).
Hence $z = G(x, y)$ is also a solution of $f(x, y, z, p, q) = 0$

(b) Attempt any Two of the following. (12)

(i) Find a partial differential equation satisfied by $z = f\left(\frac{y}{x}\right)$ where f is real valued function on $\mathbb{R} \times \mathbb{R}$.

Solution: Let $v = \frac{y}{x}$. So $\frac{\partial v}{\partial x} = -\frac{y}{x^2}, \frac{\partial v}{\partial y} = \frac{1}{x}$.
The partial differential is given by

$$\frac{\partial(z, v)}{\partial(x, y)} = 0$$

That is, $v_y\ p - v_x\ q = 0$
Substituting we get, $\frac{1}{x}\ p - \left(-\frac{y}{x^2}\ q\right) = 0$. The p.d.e. is $x\ p + y\ q = 0$.

(ii) Find the singular integral of $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0$ using the three equations : $f(x, y, z, p, q) = 0, f_p(x, y, z, p, q) = 0, f_q(x, y, z, p, q) = 0$.

Solution: We know that singular integral satisfies

$$\left.\begin{aligned} f(x, y, z, p, q) &= 0, \\ f_p(x, y, z, p, q) &= 0, \\ f_q(x, y, z, p, q) &= 0. \end{aligned}\right\}$$

$$\left.\begin{aligned} z - px - qy - p^2 - q^2 &= 0, \\ -x - 2p &= 0, \\ -y - 2q &= 0. \end{aligned}\right\}$$

This implies $p = -\frac{x}{2}, q = -\frac{y}{2}$.

Hence the singular solution is

$$\begin{aligned} & z - px - qy - p^2 - q^2 = 0 \\ \implies & z - x\frac{-x}{2} - y\frac{-y}{2} - \frac{x^2}{4} - \frac{y^2}{4} = 0 \\ \implies & z + \frac{x^2}{4} + \frac{y^2}{4} = 0 \\ \implies & 4z = -(x+y)^2 \end{aligned}$$

(iii) Solve the Lagrange's partial differential equation $\tan x\ p + \tan y\ q = \tan z$.

Solution: $\tan x\ p + \tan y\ q = \tan z$

Lagrange's auxiliary eqns are

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \quad (1).$

Taking the first two fractions, we get

$\frac{dx}{\tan x} = \frac{dy}{\tan y}$.

$\frac{\sin x}{\sin y} = c_1 \qquad (2)$

Taking the first and the last from (1).

$\frac{dx}{\tan x} = \frac{dz}{\tan x}$.

$\frac{\sin x}{\sin z} = c_2 \qquad (3)$

From (2) and (3), the required general solution is

$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin x}{\sin z}\right) = 0$

Another form of the general integral is

$\frac{\sin x}{\sin z} = \phi\left(\frac{\sin x}{\sin y}\right)$

2. (a) Attempt any One of the following: (8)

(i) (I) If $p = \phi(x, y, z)$ and $q = \psi(x, y, z)$ are obtained by solving $f(x, y, z, p, q) = 0, g(x, y, z, p, q) = 0$ for p and q then state the necessary and sufficient condition for the equation $dz = \phi(x, y, z)\ dx + \psi(x, y, z)\ dy$

to be integrable.

(II) Show that the first order partial differential equations $p = M(x, y)$ and $q = N(x, y)$ are compatible if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Solution:

$$[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(z,p)}\ p + \frac{\partial(f,g)}{\partial(y,q)} + \frac{\partial(f,g)}{\partial(z,q)}\ q = 0.$$

$p = M(x, y)$ and $q = N(x, y)$

We write these equations in the form $f = 0$ and $g = 0$.

Therefore the two equations are $f = p - M(x, y) = 0$ and $g = q - N(x, y) = 0$.

First we will show that $\dfrac{\partial(f,g)}{\partial(x,p)} + p\ \dfrac{\partial(f,g)}{\partial(z,p)} + \dfrac{\partial(f,g)}{\partial(y,q)} + q\ \dfrac{\partial(f,g)}{\partial(z,q)} = N_x - M_y$

$$\begin{aligned}
\frac{\partial(f,g)}{\partial(x,p)} &= \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix} \\
&= \begin{vmatrix} M_x & 1 \\ -N_x & 0 \end{vmatrix} \\
&= N_x \\
\frac{\partial(f,g)}{\partial(z,p)} &= \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix} \\
&= \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \\
&= 0 \\
\frac{\partial(f,g)}{\partial(y,q)} &= \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix} \\
&= \begin{vmatrix} -M_y & 0 \\ -N_y & 1 \end{vmatrix} \\
&= -My \\
\frac{\partial(f,g)}{\partial(z,q)} &= \begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix} \\
&= \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \\
&= 0
\end{aligned}$$

Hence $\dfrac{\partial(f,g)}{\partial(x,p)} + p\ \dfrac{\partial(f,g)}{\partial(z,p)} + \dfrac{\partial(f,g)}{\partial(y,q)} + q\ \dfrac{\partial(f,g)}{\partial(z,q)} = N_x + p * 0 - M_y + q * 0 =$

$N_x - M_y \qquad (*)$
Now, Suppose the first order partial differential equations $p = M(x,y)$ and $q = N(x,y)$ are compatible.
Therefore $f = p - M(x,y) = 0$ and $g = q - N(x,y) = 0$ are compatible.
Hence $\frac{\partial(f,g)}{\partial(x,p)} + p\,\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\,\frac{\partial(f,g)}{\partial(z,q)} = 0$
Hence $N_x - M_y = 0$. Thus $N_x = M_y$. That is, Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Conversely, suppose $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
To show that $f = p - M(x,y) = 0$ and $g = q - N(x,y) = 0$ are compatible.

- To show that $\frac{\partial(f,g)}{\partial(p,q)} \neq 0$.

$$\begin{aligned}\frac{\partial(f,g)}{\partial(p,q)} &= \begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ &= 1 \neq 0 \text{ on any domain } D.\end{aligned}$$

- To show that $\frac{\partial(f,g)}{\partial(x,p)} + p\,\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\,\frac{\partial(f,g)}{\partial(z,q)} = 0$

$$\begin{aligned}\frac{\partial(f,g)}{\partial(x,p)} + p\,\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\,\frac{\partial(f,g)}{\partial(z,q)} &= N_x - M_y \quad \text{from } * \\ & \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \\ &= 0\end{aligned}$$

Hence $f = 0$ and $g = 0$ are compatible.

(ii) State Charpit's Auxiliary equations and explain the Charpit's method to find a complete integral of a given partial differential equation $f(x,y,z,p,q) = 0$.

Solution:

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = -\frac{dp}{f_x + pf_z} = -\frac{dq}{f_y + qf_z} = \frac{dz}{pf_p + qf_q} = \frac{dg}{0} \qquad (7)$$

Since any of the integrals of 7 will satisfy **??**, we will find an integral of

7 which involves p or q or both. Thus we will get the required equation $g(x, y, z, p, q) = 0$

The equation 7 is called **Charpit's Auxiliary equations**.
Note that we want the functions $p = \phi$ and $q = \psi$. We can find $p = \phi$ (or $q = \psi$) from the Charpit's Auxiliary equations and then substitute the value of $p = \phi$ (or $q = \psi$) in $f = 0$ to get $q = \psi$ (or $p = \phi$). Working Rules while using Charpit's Method

STEP 1 Write the given partial differential equation in the form $f = 0$.

STEP 2 Find all the values required in Charpit's Auxiliary equation.

STEP 3 Write the Charpits' Auxiliary equation using the above values.

STEP 4 Select two proper fractions so that we can easily find the integral $g = 0$ involving at least one of p and q.

STEP 5 Solve the equations $f = 0$ and $g = 0$ for p and q and get the functions $p = \phi$ and $q = \psi$.

STEP 6 Write the Pfaffian differential equation $dz = \phi\ dx + \psi\ dy$.

STEP 7 Integral of this equation is our required complete integral of $f = 0$.

(b) Attempt any Two of the following. (12)

(i) Solve the compatible partial differential equations $xp - yq = x$ and $x^2p + q = xz$ and find a common solution.

Solution:

STEP 1 Solve the two equations for p and q.
Here $xp = x + yq$ substitue in $x^2p + q = xz \Longrightarrow x(x + yq) + q = xz \Longrightarrow x^2 + xyq + q = xz$. This means $q(xy + 1) = x(z - x)$. So, $q = \dfrac{x(z-x)}{xy+1}$.
This implies $xp = x + \dfrac{xy(z-x)}{xy+1} \Longrightarrow p = 1 + \dfrac{y(z-x)}{xy+1} \Longrightarrow p = \dfrac{1+zy}{1+xy}$

STEP 2 Construct the Pfaffian differential equation $dz = \phi\ dx + \psi\ dy$.
That is, $dz = \dfrac{1+zy}{1+xy}\ dx + \dfrac{x(z-x)}{1+xy}\ dy \Longrightarrow z\ dz = d(xy) \Longrightarrow z^2 =$

$2xy + c$

$$
\begin{aligned}
(1 + xy)\ dz &= (1 + zy)\ dx + x(z - x)\ dy \\
(1 + xy)\ dz &= dx + zy\ dx + xz\ dy - x^2\ dy \\
(1 + xy)\ dz &= dx + z(y\ dx + z\ dy) - x^2\ dy \\
(1 + xy)\ dz - z(y\ dx + z\ dy) &= dx - x^2\ dy \\
\frac{(1 + xy)\ dz - z(y\ dx + z\ dy)}{(1 + xy)^2} &= \frac{dx - x^2\ dy}{(1 + xy)^2} \\
d\left(\frac{z}{(1 + xy)}\right) &= \frac{\frac{1}{x^2}dx -\ dy}{\left(\frac{1}{x} + y\right)^2} \\
d\left(\frac{z}{(1 + xy)}\right) &= -\frac{-\frac{1}{x^2}dx +\ dy}{\left(\frac{1}{x} + y\right)^2} \\
d\left(\frac{z}{(1 + xy)}\right) &= d\left(\frac{1}{\frac{1}{x} + y}\right)
\end{aligned}
$$

On integrating, we get,

$\dfrac{z}{(1 + xy)} = \dfrac{1}{\frac{1}{x} + y} + c.$

$\dfrac{z}{(1 + xy)} = \dfrac{x}{1 + xy} + c.$

STEP 3 $\dfrac{z}{(1 + xy)} = \dfrac{x}{1 + xy} + c$ is the required solution.

(ii) Show that $dz = \phi(x, y, z)\ dx + \psi(x, y, z)\ dy$ is integrable if and only if

$$-\phi\ \psi_z + \psi\ \phi_z - \psi_x + \phi_y = 0.$$

Solution: where $\overline{\mathbf{X}} = (\phi(x,y,z), \psi(x,y,z), -1)$.

$$\text{curl } \overline{\mathbf{X}} = \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}$$

$$= \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi & \psi & -1 \end{pmatrix}$$

$$= \Big(\frac{\partial}{\partial y}\ (-1) - \frac{\partial}{\partial z}\ \psi\Big)\hat{\mathbf{i}} - \Big(\frac{\partial}{\partial x}\ (-1) - \frac{\partial}{\partial z}\ \phi\Big)\hat{\mathbf{j}} + \Big(\frac{\partial}{\partial x}\ \psi - \frac{\partial}{\partial y}\ \phi\Big)\hat{\mathbf{k}}$$

$$= -\psi_z\ \hat{\mathbf{i}} + \phi_z\ \hat{\mathbf{j}} + (\psi_x - \phi_y)\ \hat{\mathbf{k}}$$

$$\begin{aligned} &= (-\psi_z,\ \phi_z,\ \psi_x - \phi_y) \\ \overline{\mathbf{X}} \cdot \text{curl } \overline{\mathbf{X}} &= (\phi,\ \psi,\ -1) \cdot (-\psi_z,\ \phi_z,\ \psi_x - \phi_y) \\ &= -\phi\ \psi_z + \psi\ \phi_z - \psi_x + \phi_y \end{aligned}$$

Hence $dz = \phi(x,y,z)\ dx + \psi(x,y,z)\ dy$ is integrable if and only if $-\phi\ \psi_z + \psi\ \phi_z - \psi_x + \phi_y = 0$.

(iii) Find a complete integral of $x^2p^2 + y^2q^2 - 4 = 0$.

Solution: We find all the values required in Charpit's Auxiliary equation. Charpit's Auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{(p\ f_p + q\ f_q)} = -\frac{dp}{(f_x + pf_z)} = -\frac{dq}{(f_y + qf_z)}$$

f_p	f_q	$pf_p + qf_q$	$f_x + pf_z$	$f_y + qf_z$
$2x^2p$	$2y^2q$	$p(2x^2p) + q(2y^2q)$	$2xp^2 + p(0)$	$2yq^2 + q(0)$
$2x^2p$	$2y^2q$	$2x^2p^2 + 2y^2q^2$	$2xp^2$	$2yq^2$

$$\frac{dx}{2x^2p} = \frac{dy}{2y^2q} = \frac{dz}{2(x^2p^2 + y^2q^2)} = -\frac{dp}{2xp^2} = -\frac{dq}{2yq^2}$$

Consider first and fourth ratio.

$$\frac{dx}{2x^2p} = -\frac{dp}{2xp^2}$$
$$\frac{dx}{x} = -\frac{dp}{p}$$

Integrating both sides, we get, $\ln x = -\ln p + \ln a$. This means $xp = a$. That is, $p = \frac{a}{x}$.

Substituting this in $f = x^2p^2 + y^2q^2 - 4 = 0$, we get, $x^2\frac{a^2}{x^2} + y^2q^2 - 4 = 0$.

$$\begin{aligned} x^2\frac{a^2}{x^2} + y^2q^2 - 4 &= 0 \\ a^2 + y^2q^2 &= 4 \\ q &= \frac{\sqrt{4-a^2}}{y} \end{aligned}$$

Consider the Pfaffian differential equation $dz = p\ dx + q\ dy$.

$$dz = \frac{a}{x}\ dx + \frac{\sqrt{4-a^2}}{y}\ dy$$
$$\implies z = a\ln x + \sqrt{4-a^2}\ln y + b$$

$z = F(x, y; a, b) = a\ln x + \sqrt{4-a^2}\ln y + b$ is a complete integral of $f = x^2p^2 + y^2q^2 - 4 = 0$

Note: If we consider second and fourth ratio, that is, $\frac{dy}{2y^2q} = -\frac{dq}{2yq^2}$ then we get $z = c\ln y + \sqrt{4-c^2}\ln x + d$ as a complete integral.

So, $z = a \ln x + \sqrt{4 - a^2}\ \ln y + b$
and $z = c \ln y + \sqrt{4 - c^2}\ \ln x + d$
are complete integrals of $f = x^2p^2 + y^2q^2 - 4 = 0$.

3. (a) Attempt any One of the following. (8)

(i) Consider the quasi-linear equation $P(x, y, z)\ p+Q(x, y, z)\ q = R(x, y, z)$ where P, Q and R are continuously differentiable functions on a domain $\Omega \subseteq \mathbb{R}^3$. If $S : z = u(x, y)$ is the surface obtained by taking the union of characteristic curves of the given p.d.e. where $u(x, y)$ is a continuously differentiable function then prove that S is the integral surface of the p.d.e.

Solution: We want to show that S is an integral surface.
Let $A(x_0, y_0, u(x_0, y_0))$ be any point on S.
We want to show that A satisfies the PDE $P(x, y, z)\ p + Q(x, y, z)\ q = R(x, y, z)$.
This means we want to show that

$$P(x_0, y_0, u(x_0, y_0))\ u_x(x_0, y_0) + Q(x_0, y_0, u(x_0, y_0))\ u_y(x_0, y_0) = R(x_0, y_0, u(x_0, y_0)).$$

Since S is a union of characteristic curves, there is a characteristic curve γ_A passing through A that lies on S (curve is lying on S).
Since normal to S at A is in the direction $(u_x(x_0, y_0), u_y(x_0, y_0), -1)$ and the tangent to the curve γ_A at A is in the direction $\Big(P(x_0, y_0, u(x_0, y_0)), Q(x_0, y_0, u(x_0, y_0)), R(x$
the dot product of the two is 0.
Hence

$$\Big(u_x(x_0, y_0), u_y(x_0, y_0), -1\Big) \cdot \Big(P(x_0, y_0, u(x_0, y_0)), Q(x_0, y_0, u(x_0, y_0)), R(x_0, y_0, u(x_0, y_0))\Big) =$$

Thus

$$P(x_0, y_0, u(x_0, y_0))u_x(x_0, y_0) + Q(x_0, y_0, u(x_0, y_0))u_y(x_0, y_0) + R(x_0, y_0, u(x_0, y_0))(-1) = 0.$$

Hence $A(x_0, y_0, u(x_0, y_0))$ satisfies the PDE $P(x, y, z)\ p + Q(x, y, z)\ q = R$ where $p = z_x = u_x, q = z_y = u_y$.

(ii) Write a short note on the characteristic strip and characteristic curve for a non linear first order partial differential equation $f(x, y, z, p, q) = 0$.

Solution: The characteristic differential equations related to the p.d.e. $f(x, y, z, p, q) = 0$ are

$$\left.\begin{aligned} \frac{dx}{dt} &= f_p, \\ \frac{dy}{dt} &= f_q, \\ \frac{dz}{dt} &= pf_p + qf_q, \\ \frac{dp}{dt} &= -f_x - f_z * p, \\ \frac{dq}{dt} &= -f_y - f_z * q, \end{aligned}\right\}$$

A solution $(x(t), y(t), z(t), p(t), q(t))$ of the above system of ordinary differential equations can be interpreted as a strip. The first three functions $(x(t), y(t), z(t))$ determine a space curve. At each point of this space curve, $p(t)$ and $q(t)$ define a tangent plane with $(p, q, -1)$ as the normal vector. The curve along with these tangent planes at each point is called a characteristic strip and the curve is called the characteristic curve.

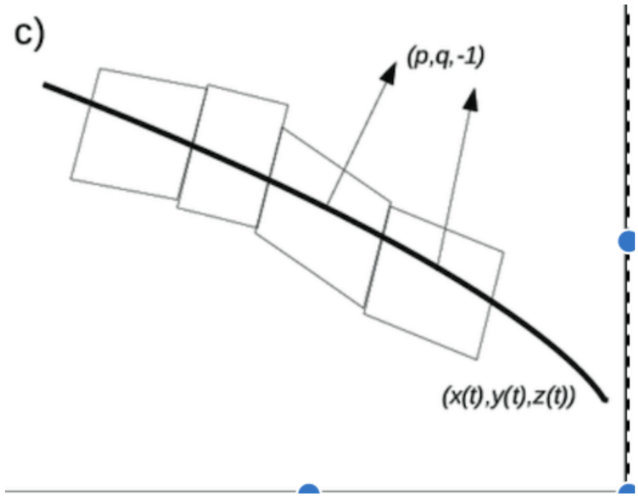

Figure 1: Characteristic strip

Any set of five functions $x(t), y(t), z(t), p(t)$ and $q(t)$ of the above system can be interpreted as a strip only if the following condition called the **strip condition**

$$\frac{dz}{dt} = p(t)\frac{dx}{dt} + q(t)\frac{dy}{dt} \tag{8}$$

is satisfied.

(b) Attempt any Two of the following. (12)

(i) Find the solution of the initial value problem for the quasi-linear equation $p - z\,q = -z$ for all y and $x > 0$ with the initial data curve $C : x_0(s) = 0, y_0(s) = s, z_0(s) = -2s, \quad -\infty < s < \infty$.

Solution: Here $P(x, y, z) = 1, Q(x, y, z) = -z, R(x, y, z) = -z$ and $x_0(s) = 0, y_0(s) = s, z_0(s) = -2s$.
We will show that

$$\frac{dy_0}{ds}P\Big(x_0(s), y_0(s), z_0(s)\Big) - \frac{dx_0}{ds}Q\Big(x_0(s), y_0(s), z_0(s)\Big) \neq 0 \text{ on } (-\infty,\ \infty).$$

$$\frac{dy_0}{ds}P\Big(x_0(s), y_0(s), z_0(s)\Big) - \frac{dx_0}{ds}Q\Big(x_0(s), y_0(s), z_0(s)\Big) = (1)(1) - (0)\Big(2s\Big)$$
$$= 1 \neq 0 \quad \text{for } -\infty < s < \infty.$$

Now we will solve the following system $\begin{cases} \dfrac{dx}{dt} = 1 \\ \dfrac{dy}{dt} = -z \\ \dfrac{dz}{dt} = -z. \end{cases}$

with initial conditions, $x(s,0) = 0, y(s,0) = s, z(s,0) = -2s$ at $t = 0$.
The family of characteristic curves through the initial data curve is

$$\frac{dx}{dt} = 1 \Longrightarrow x = t + c_1 \Longrightarrow x(s,o) = 0 + c_1 \Longrightarrow c_1 = 0$$

$$\Longrightarrow x = t \qquad (1)$$

$$\frac{dz}{dt} = -z \Longrightarrow \ln z = -t + \ln c_2 \Longrightarrow e^{-t} = \frac{z}{c_2} \Longrightarrow e^0 = \frac{-2s}{c_2} \Longrightarrow c_2 = -2s$$

$$\Longrightarrow z = -2se^{-t} \qquad (2)$$

$$\frac{dy}{dt} = -z \Longrightarrow \frac{dy}{dt} = 2se^{-t} \Longrightarrow y = -2se^{-t} + c_3 \Longrightarrow y(s,o) = -2s + c_3 \Longrightarrow c_3 = s + 2s = 3$$

$$\Longrightarrow y = -2se^{-t} + 3s \qquad (3)$$

We we will solve(1) and (3) for s and t in terms of x and y.

$$x = t \Longrightarrow y = -2se^{-x} + 3s \Longrightarrow y = s(-2e^{-x} + 3) \Longrightarrow s = \frac{y}{3 - 2e^{-x}}$$

Now, we will substitute these values in $z = -2se^{-t} = -\dfrac{2ye^{-x}}{3 - 2e^{-x}} = \dfrac{2y}{2 - 3e^{x}}$.
The solution breaks down at $e^x = \dfrac{2}{3}$ that is, the solution breaks down at $x = \ln \dfrac{2}{3}$.

(ii) Find the initial strip for $z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y)$ passing through the $x-$axis.

Solution: Solution is passing through the $x-$axis.
Hence the initial data curve is $x = x_0(s) = s, y = y_0(s) = 0, z = z_0(s) = 0$.

Initial strip condition is

$$\frac{dz_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds}$$

$$\implies 0 = p_0 * 1 + q_0 * 0$$

$$\implies p_0 = 0$$

Substitute $x = x_0(s) = s, y = y_0(s) = 0, z = z_0(s) = 0, p_0 = 0$ in the given equation.

$$0 = \frac{1}{2}(0 + q_0^2) + (0 - s)(q_0 - 0)$$

$$\implies \frac{q_0^2}{2} - q_0 * s = 0$$

$$\implies q_0^2 - 2q_0 * s = 0$$

$$\implies q_0(q_0 - 2s) = 0$$

$$\implies q_0 = 0 \quad \text{or} \quad q_0 = 2s$$

Therefore there are two initial strips:
Case 1: $x = x_0(s) = s, y = y_0(s) = 0, z = z_0(s) = 0, p = p_0(s) = 0, q = q_0(s) = 0$ and
Case 2: $x = x_0(s) = s, y = y_0(s) = 0, z = z_0(s) = 0, p = p_0(s) = 0, q = q_0(s) = 2s$.

(iii) Find the characteristics differential equations and the characteristic strips of $pq = xy$.

Solution:
We find the characteristic differential equations.

$$
\begin{aligned}
f(x,y,z,p,q) = 0 &\implies pq - xy = 0 \\
\implies f(x,y,z,p,q) &= pq - xy \\
\frac{dx}{dt} &= f_p = q \\
\frac{dy}{dt} &= f_q = p
\end{aligned}
$$

$$
\begin{aligned}
\frac{dz}{dt} &= pf_p + qf_q = 2pq \\
\frac{dp}{dt} &= -f_x - f_z * p = y \\
\frac{dq}{dt} &= -f_y - f_z * q = x
\end{aligned}
$$

We find the characteristic strips. We consider $\frac{dx}{dt} = q$ and $\frac{dq}{dt} = x$.
We write as $\frac{dx}{dt} = 0 * x + 1 * q$ and $\frac{dq}{dt} = 1 * x + 0 * q$.
Auxiliary equation for the system $\frac{dx}{dt} = a_1 x + b_1 y$ and $\frac{dy}{dt} = a_2 x + b_2 y$ is
$m^2 - (a_1 + b_2)m + a_1 b_2 - a_2 b_1 = m^2 - 0.m - 1 = 0$.
This means $m^2 = 1$ and hence $m = \pm 1$.
If $m = 1, x = A_1 e^t, q = B_1 e^t$.
To find values of A and B.
$(a_1 - m)A_1 + b_1 B_1 = 0 \implies (0 - 1)A_1 + 1 * B_1 = 0$.
That is, $A_1 = B_1$. Hence we choose $A_1 = B_1 = 1$.
One solution is $x = e^t, q = e^t$.
If $m = -1, x = A_2 e^{-t}, q = B_2 e^{-t}$.
To find values of A_2 and B_2.
$(a_1 - m)A_2 + b_1 B_2 = 0 \implies (0 + 1)A_2 + 1 * B_2 = 0$.
That is, $A_2 = -B_2$. Hence we choose $A_2 = 1, B_2 = -1$.
Second independent solution is $x = e^{-t}, q = -e^{-t}$.
Hence general solution $\boldsymbol{x = Ae^t + Be^{-t}, q = Ae^t - Be^{-t}}$.
Similarly, We consider $\frac{dy}{dt} = p$ and $\frac{dp}{dt} = y \implies \boldsymbol{y = Ce^t + De^{-t}, p = Ce^t - De^{-t}}$.

Substituting in $\frac{dz}{dt}$, we get

$$\begin{aligned}\frac{dz}{dt} &= 2pq = 2(Ce^t - De^{-t})(Ae^t - Be^{-t}) \\ &= 2(Ce^t - De^{-t})(Ae^t - Be^{-t}) \\ &= 2\Big(ACe^{2t} + BDe^{-2t} - BC - AD\Big) \\ z &= ACe^{2t} - BDe^{-2t} - 2(BC + AD)t + C\end{aligned}$$

Hence the characteristic strips are

$$\begin{aligned}x &= Ae^t + Be^{-t}, \\ y &= Ce^t + De^{-t}, \\ p &= Ce^t - De^{-t}, \\ q &= Ae^t - Be^{-t}, \\ z &= ACe^{2t} - BDe^{-2t} - 2(BC + AD)t + C\end{aligned}$$

4. Attempt any Three of the following. (15)

(a) Find a partial differential equation satisfied by $z = f(x^2 + y^2)$.

Solution: Let $v = x^2 + y^2$. So $\frac{\partial v}{\partial x} = 2x, \frac{\partial v}{\partial y} = 2y$.
The partial differential is given by

$$\frac{\partial(z, v)}{\partial(x, y)} = 0$$

That is, $v_y\ p - v_x\ q = 0$
Substituting we get, $(2y)\ p - (2x)\ q = 0$. The p.d.e. is $y\ p - x\ q = 0$

(b) Show that $(x - a)^2 + (y - b)^2 + z^2 = 1$ is a solution of $z^2(1 + p^2 + q^2) = 1$.

Solution:

$$(x - a)^2 + (y - b)^2 + z^2 = 1$$

Differentiation partially with respect to x and y we get

$2(x-a)+2zz_x=0$
$\implies zz_x=-(x-a)$
$\implies zp=-(x-a)$
$\implies z^2p^2=(x-a)^2$

$2(y-b)+2zz_y=0$
$\implies zz_y=-(y-b)$
$\implies zq=-(y-b)$
$\implies z^2q^2=(y-b)^2$

$$\begin{aligned} z^2(1+p^2+q^2) &= z^2+z^2p^2+z^2q^2 \\ &= z^2+(x-a)^2+(y-b)^2 \\ &= 1 \end{aligned}$$

Hence $(x-a)^2+(y-b)^2+z^2=1$ is a solution of (1).

(c) Show that the partial differential equations $xp=yq$ and $z(xp+yq)=2xy$ are compatible.

Solution:

- To show that $\dfrac{\partial(f,g)}{\partial(p,q)}\neq 0$.

$$\begin{aligned} \frac{\partial(f,g)}{\partial(p,q)} &= \begin{vmatrix} f_p & f_q \\ g_p & g_q \end{vmatrix} \\ &= \begin{vmatrix} x & -y \\ zx & zy \end{vmatrix} \\ &= 2xyz \neq 0 \text{ on } D=\{(x,y,z)\in\mathbb{R}^3 : xyz\neq 0\} \end{aligned}$$

- To show that $\dfrac{\partial(f,g)}{\partial(x,p)}+p\,\dfrac{\partial(f,g)}{\partial(z,p)}+\dfrac{\partial(f,g)}{\partial(y,q)}+q\,\dfrac{\partial(f,g)}{\partial(z,q)}=0$

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix}$$
$$= \begin{vmatrix} p & x \\ zp-2y & zx \end{vmatrix}$$
$$= zpx - zpx + 2xy = 2xy$$
$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix}$$
$$= \begin{vmatrix} 0 & x \\ xp+yq & zx \end{vmatrix}$$
$$= -x^2p - xyq$$
$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix}$$
$$= \begin{vmatrix} -q & -y \\ zq-2x & zy \end{vmatrix}$$
$$= -2xy$$
$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix}$$
$$= \begin{vmatrix} 0 & -y \\ (xp+yq) & zy \end{vmatrix}$$
$$= y(xp+yq)$$

Hence $\dfrac{\partial(f,g)}{\partial(x,p)} + p\,\dfrac{\partial(f,g)}{\partial(z,p)} + \dfrac{\partial(f,g)}{\partial(y,q)} + q\,\dfrac{\partial(f,g)}{\partial(z,q)}$
$= 2xy + p * (-x^2p - xyq) - 2xy + q * y(xp+yq)$
$= -p^2x^2 + y^2q^2$
$= 0$ as $xp = yq$

Hence the two equations are compatible.

(d) Find a complete integral of the partial differential equation $z = px + qy + pq$.

Solution: If the given partial differential equation is of the form $z = xp + yq + h(p,q)$ then $z = ax + by + h(a,b)$ is its complete integral.
hence here $z = ax + by + ab$.

(e) For the partial differential equation $x^3\,p + y(3x^2+y)\,q = z(2x^2+y)$ and the initial data curve $x_0(s) = 1, y_0(s) = s, z_0(s) = s^2 + s$, find the value of

$$\frac{dy_0}{ds} P\Big(x_0(s), y_0(s), z_0(s)\Big) - \frac{dx_0}{ds} Q\Big(x_0(s), y_0(s), z_0(s)\Big)$$

where $x^3\ p + y(3x^2 + y)\ q = z(2x^2 + y)$ is compared with $P(x, y, z)\ p + Q(x, y, z)\ q = R(x, y, z)$.

Solution: Here $P = x^3, Q = y(3x^2 + y), R = z(2x^2 + y)$
$x = x_0(s) = 1, y = y_0(s) = s, z = z_0(s) = s^2 + s$.
We will show that

$$\frac{dy_0}{ds} P\Big(x_0(s), y_0(s), z_0(s)\Big) - \frac{dx_0}{ds} Q\Big(x_0(s), y_0(s), z_0(s)\Big) \neq 0 \quad \text{on } I.$$

$$\begin{aligned}\frac{dy_0}{ds} P\Big(x_0(s), y_0(s), z_0(s)\Big) - \frac{dx_0}{ds} Q\Big(x_0(s), y_0(s), z_0(s)\Big) &= (1)P(1, s, s^2 + s) - (0) * Q(1, s, s^2 + \\ &= 1(1)^3 = 1 \neq 0\end{aligned}$$

(f) Show that the characteristic curves of $z\ p + q = 0$ containing the initial data curve $C : x_0(s) = s, y = y_0(s) = 0, z_0(s) = f(s)$ where $f(s) = \begin{cases} 1 & \text{if } s \leq 0, \\ 1 - s & \text{if } 0 \leq s \leq 1, \\ 0 & \text{if } s \geq 1. \end{cases}$

are straight lines given by $x = y\ f(s) + s$.

Solution: Here $P(x, y, z) = z, Q(x, y, z) = 1, R(x, y, z) = 0$ and $x_0(s) = s, y_0(s) = 0, z_0(s) = f(s)$.
We will show that

$$\frac{dy_0}{ds} P\Big(x_0(s), y_0(s), z_0(s)\Big) - \frac{dx_0}{ds} Q\Big(x_0(s), y_0(s), z_0(s)\Big) \neq 0 \quad \text{on } I.$$

$$\begin{aligned}\frac{dy_0}{ds} P\Big(x_0(s), y_0(s), z_0(s)\Big) - \frac{dx_0}{ds} Q\Big(x_0(s), y_0(s), z_0(s)\Big) &= (0)P\Big(s, 0, f(s)\Big) - (1)\Big(1\Big) \\ &= -1 \neq 0\end{aligned}$$

Now we will solve the following system $\begin{cases} \dfrac{dx}{dt} = z \\ \dfrac{dy}{dt} = 1 \\ \dfrac{dz}{dt} = 0. \end{cases}$

with initial conditions, $x(s,0) = s, y(s,0) = 0, z(s,0) = f(s)$ at $t = 0$.
The family of characteristic curves through the initial data curve is

$$\frac{dy}{dt} = 1 \Longrightarrow y = t + c_1 \Longrightarrow y(s,o) = 0 + c_1 \Longrightarrow c_1 = 0$$

$$\Longrightarrow y = t \qquad\qquad (1)$$

$$\frac{dz}{dt} = 0 \Longrightarrow z = c_2 \Longrightarrow z(s,o) = c_2 \Longrightarrow c_2 = \begin{cases} 1 & \text{if } \ s \le 0, \\ 1-s & \text{if } \ 0 \le s \le 1, \\ 0 & \text{if } \ s \ge 1. \end{cases}$$

$$\Longrightarrow z = \begin{cases} 1 & \text{if } \ s \le 0, \\ 1-s & \text{if } \ 0 \le s \le 1, \\ 0 & \text{if } \ s \ge 1. \end{cases} \qquad\qquad (2)$$

$$\frac{dx}{dt} = z \Longrightarrow \frac{dx}{dt} = \begin{cases} 1 & \text{if } \ s \le 0, \\ 1-s & \text{if } \ 0 \le s \le 1, \\ 0 & \text{if } \ s \ge 1. \end{cases}$$

$$\Longrightarrow x = \begin{cases} t + c_2 & \text{if } \ s \le 0, \\ t - st + c_3 & \text{if } \ 0 \le s \le 1, \\ 0 + c_4 & \text{if } \ s \ge 1. \end{cases}$$

$$\Longrightarrow x(s,0) = s = \begin{cases} 0 + c_2 & \text{if } \ s \le 0, \\ 0 + c_3 & \text{if } \ 0 \le s \le 1, \\ 0 + c_4 & \text{if } \ s \ge 1. \end{cases}$$

$$\Longrightarrow x = \begin{cases} t + s & \text{if } \ s \le 0, \\ t - st + s & \text{if } \ 0 \le s \le 1, \\ s & \text{if } \ s \ge 1. \end{cases} \qquad\qquad (3)$$

We will show that $x = yf(s) + s$.

$$yf(s) + s = \begin{cases} y(1) + s & \text{if } s \le 0, \\ y(1-s) + s & \text{if } 0 \le s \le 1, \\ y(0) + s & \text{if } 1 \le s \end{cases}$$

Since $y = t$ from (1), we get,

$$yf(s) + s = \begin{cases} t + s & \text{if } s \leq 0, \\ t(1-s) + s & \text{if } 0 \leq s \leq 1, \\ s & \text{if } 1 \leq s. \end{cases}$$

Hence $yf(s) + s = x$.

Hence the characteristic curves are straight lines intersecting the $x-$ axis at $(s, 0)$.

xxxxxxxxx