Practical No 6 : Solve initial value problem for quasi-linear PDE

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Solve Initial Value Problem for Quasi-Linear PDE.

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Problem 1:

Example 1

Find the general solution of the quasilinear PDE

$$a\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial t}=c$$

where

$$b=1, \qquad c=0$$

Using equation (9b), we have the reciprocal of the slope of characteristic curves

$$\frac{dx}{dt}=\frac{a}{b}=a$$

Separating the variables and integrating to obtain

$$x=at+A$$

where A is an arbitrary constant. Further, from (9a), we have

$$\frac{du}{dt}=\frac{c}{b}=0$$

$$\frac{du}{dt} = \frac{c}{b} = 0$$

which leads to

$$u = B$$

where B is an arbitrary constant. Thus,

$$x - at = A, \qquad u = B$$

is a two-parameter family of characteristic curves. Specifying B as a function of A defines a one-parameter family of characteristic curves, a solution surface. Thus, the general solution is expressed by writing $B = f(A)$. Therefore, the general solution is

$$u(x,y) = f(x - at)$$

where $f(\cdot)$ is an arbitrary function. By direct substitution, it is east to see that we do indeed have a solution of the equation for arbitrary f.

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Problem 2:

Example 2

Find the general solution of the quasilinear PDE

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

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Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

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where

$$a = x, \qquad b = y, \qquad c = u$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}$$

Using the first of Lagrange–Charpit equation, we have the reciprocal of the slope of characteristic curves

$$\frac{dx}{dy} = \frac{x}{y}$$

Separating the variables and integrating to obtain

$$\ln x = \ln y + \ln A \qquad \Longrightarrow \qquad x = Ay$$

where A is an arbitrary constant. Further, we have

$$\frac{du}{dy} = \frac{u}{y}$$

Separating the variables and integrating to obtain

$$\ln u = \ln y + \ln B \qquad \Longrightarrow \qquad u = By$$

where B is an arbitrary constant. Thus,

$$\frac{x}{y} = A, \qquad \frac{u}{y} = B$$

is a two-parameter family of characteristic curves. Specifying B as a function of A defines a one-parameter family of characteristic curves, a solution surface. Thus, the general solution is expressed by writing $B = f(A)$. Therefore, the general solution is

$$u(x,y) = yf\left(\frac{x}{y}\right)$$

where $f(\cdot)$ is an arbitrary function. By direct substitution, it is east to see that we do indeed have a solution of the equation for arbitrary f.

It may be noted that had we selected the equations of the form

$$\frac{dy}{dx} = \frac{y}{x}, \qquad \frac{du}{dx} = \frac{u}{x}$$

we would have obtained the general solution as

$$u(x,y) = xg\left(\frac{y}{x}\right)$$

where $g(\cdot)$ is an arbitrary function

Problem 3:

Example 3

Find the general solution of the quasilinear PDE

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} = (x+y)u$$

Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

where

$$a = x^2, \qquad b = y^2, \qquad c = (x+y)u$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{(x+y)u}$$

Using the first of Lagrange–Charpit equation, we have the reciprocal of the slope of characteristic curves

$$\frac{dx}{dy} = \frac{x^2}{y^2}$$

Separating the variables and integrating to obtain

$$x^{-1} = y^{-1} + A'$$

where A' is an arbitrary constant. In addition, when we subtract the equation $dy = y^2 \dfrac{du}{(x+y)u}$ from $dx = x^2 \dfrac{du}{(x+y)u}$ to obtain

$$dx - dy = (x^2 - y^2)\frac{du}{(x+y)u} \quad \Longrightarrow \quad \frac{dx-dy}{x-y} = \frac{du}{u} \quad \Longrightarrow \quad u = B(x-y)$$

where B is an arbitrary constant. Thus,

$$\frac{xy}{x-y} = A, \qquad \frac{u}{x-y} = B$$

is a two-parameter family of characteristic curves. Specifying B as a function of A defines a one-parameter family of characteristic curves, a solution surface. Thus, the general solution is expressed by writing $B = f(A)$. Therefore, the general solution is

$$u(x,y) = (x-y)f\left(\frac{xy}{x-y}\right)$$

where $f(\cdot)$ is an arbitrary function.

Problem 4:

Example 4

Find the general solution of the Cauchy problem governed by the quasilinear PDE

$$2y\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial y} = 2yu^2$$

Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

where

$$a = 2y, \qquad b = u, \qquad c = 2yu^2$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{2y} = \frac{dy}{u} = \frac{du}{2yu^2}$$

Using the first of Lagrange–Charpit equation, we have the reciprocal of the slope of charac
curves

$$\frac{dx}{dy} = \frac{2y}{u}$$

Further, we have

$$\frac{du}{dy} = \frac{2yu^2}{u} = 2yu \quad (\text{for } u \neq 0)$$

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Separating the variables and integrating to obtain

$$\int \frac{du}{u} = 2\int y\,dy + \ln B \qquad \Longrightarrow \qquad \ln u = y^2 + \ln B$$

where B is an arbitrary constant. The above equation can be rewritten as

$$u = Be^{y^2}$$

Plugging the value of u in the expression for dx/dy yields

$$\frac{dx}{dy} = \frac{2y}{Be^{y^2}}$$

which on integration

$$B\int dx = \int 2ye^{-y^2}dy + A \qquad \Longrightarrow \qquad Bx = -e^{-y^2} + A$$

where A is the second arbitrary constant. Thus,

$$Bx + e^{-y^2} = A, \qquad u = Be^{y^2}$$

is a two-parameter family of characteristic curves. Specifying B as a function of A defines a one-parameter family of characteristic curves, a solution surface. Thus, the general solution is expressed by writing $B = f(A)$, where

$$A = Bx + e^{-y^2} = ue^{-y^2}x + e^{-y^2} = (xu+1)e^{-y^2}$$

which gives

$$u(x,y) = e^{y^2} f\left[(1+xu)e^{-y^2}\right]$$

an implicit relation for $u(x,y)$, where $f(\cdot)$ is an arbitrary function.

Problem 5:

Example 5

Find the solution of the Cauchy problem governed by the linear PDE

$$a\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

that takes on the values

$$F(x) = u(x,0) = \begin{cases} -\frac{x}{3} & \text{if } x \leq 0 \\ 2x+3 & \text{if } x > 0 \end{cases}$$

Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial t} = c$$

where

$$b = 1, \qquad c = 0$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{a} = \frac{dt}{1} = \frac{du}{0}$$

Using the Lagrange–Charpit equation, we have the reciprocal of the slope of characteristic curves

$$\frac{dx}{dt} = a$$

Separating the variables and integrating to obtain

$$x = at + A$$

where A is an arbitrary constant. Further, we have

$$\frac{du}{dt} = 0$$

which leads to

$$u = B$$

where B is an arbitrary constant. Note that for selecting the second equation above, we have two choices; they are du/dt and du/dx. We select du/dt if the Cauchy data is specified on $t = 0$ line as a function of x. On the other hand, du/dx is selected if the Cauchy data is specified on $x = 0$ line as a function of t.

Thus,

$$x - at = A, \qquad u = B$$

is a two-parameter family of characteristic curves. For solution curves to pass through the initial data, $F(x) = u(x,0) = -x/3$ for $x \leq 0$, we set

$$\xi = A, \qquad -\frac{\xi}{3} = B \qquad \Longrightarrow \qquad A = \xi, \qquad B = -\frac{\xi}{3}$$

where ξ is a constant (x-intercept, in this case) that identifies a characteristic curve. Thus, the characteristic and solution curves through this part of the initial curve are

$$x = at + \xi, \qquad u = -\frac{\xi}{3}$$

Eliminating ξ from the second equation using the first yields

$$u(x,t) = -\frac{1}{3}(x - at) = \frac{1}{3}(at - x)$$

We could also obtain the general solution first and then use the Cauchy data to obtain the particular solution. The general solution is expressed by writing $B = f(A)$ as follows

$$u(x,t) = f(x - at)$$

where $f(\cdot)$ is an arbitrary function. For solution curves to pass through the initial data, $u(x,0) = -x/3$ for $x \leq 0$, we set

$$-\frac{x}{3} = f(x) \qquad \Longrightarrow \qquad f(x - at) = -\frac{x - at}{3}$$

Therefore the solution of the PDE for $x \leq 0$ is

$$u(x,t) = f(x - at) = \frac{1}{3}(at - x)$$

For solution curves to pass through the initial data, $F(x) = u(x,0) = 2x + 3$ for $x > 0$, we set

$$\xi = A, \qquad 2\xi + 3 = B \qquad \Longrightarrow \qquad A = \xi, \qquad B = 2\xi + 3$$

Thus, the characteristic and solution curves through this part of the initial curve are

$$x = at + \xi, \qquad u = 2\xi + 3$$

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Eliminating ξ from the second equation using the first yields

$$u(x,t) = 2(x-at)+3$$

Again, we could obtain the general solution first and then use the Cauchy data to obtain the particular solution. The general solution is expressed by writing $B = f(A)$ as follows

$$u(x,t) = f(x-at)$$

where $f(\cdot)$ is an arbitrary function. For solution curves to pass through the initial data, $u(x,0) = 2x+3$ for $x > 0$, we set

$$2x+3 = f(x) \qquad \Longrightarrow \qquad f(x-at) = 2(x-at)+3$$

Therefore the solution of the PDE for $x > 0$ is

$$u(x,t) = f(x-at) = 2(x-at)+3$$

The solution surface is composed of two planes, and to determine regions in the (x,t) plane onto which these planes project, we draw base characteristic curves. They are the lines $x = at+\xi$ shown in Figure 1.10a. Below the characteristic curve $x = at$ are characteristic curves along which $u = -\xi/3$; along characteristic curves above $x = at$, $u = 2\xi+1$. The solution surface in Figure 1.10b, consists of two planes above the regions corresponding to these two sets of characteristic curves. It is discontinuous along the base characteristic curve $x = at$ through the point $(0,0)$ where the initial data is discontinuous.

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Problem 6:

Example 6

Find the solution surface for the linear PDE

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$$

subject to the Cauchy condition that $u = \sin x$ on $y = 3x + 1$.

Solution The Cauchy condition in this case is prescribed on an oblique straight line C. The characteristic equations of the PDE in nonparametric form is given by

$$\frac{dx}{dy} = \frac{1}{2}$$

$$\frac{du}{dy} = 0$$

These equations are now solved to get the equation of characteristic curves. Integrating gives

$$x = \frac{y}{2} + A, \qquad u = B$$

where A and B are arbitrary constants that identifies the characteristics. In fact, A is the x-intercept (denoted by ξ) of the characteristics. The y-intercept of the characteristic line η is then equal to 2ξ. Therefore,

$$x = \frac{y}{2} + \xi, \qquad u = B$$

For solution curves to pass through the initial data $u = \sin x$ on $y = 3x + 1$, we set

$$x = \frac{3x+1}{2} + \xi, \qquad \sin x = B$$

which leads to

$$x = -2\xi - 1, \qquad B = \sin(-2\xi - 1)$$

on $y = 3x + 1$ line. Thus the characteristic and solution curves through the initial curve are

$$x = \frac{y}{2} + \xi, \qquad u = \sin(-2\xi - 1)$$

Eliminating ξ from the second equation using the first yields

$$u(x,y) = \sin(y - 2x - 1)$$

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Differentiation confirms that $u(x,y)$ satisfies the PDE, and it also satisfies the Cauchy condition on the line $y=3x+1$, so it is the required solution.

We could have also obtained the general solution first and then use the Cauchy data to obtain the particular solution. The general solution is expressed by writing $B=f(A)$ as follows

$$u(x,y) = f\left(x-\frac{y}{2}\right)$$

where $f(\cdot)$ is an arbitrary function. For solution curves to pass through the initial data $u=\sin x$ on $y=3x+1$, we set

$$\sin x = f\left(\frac{-x-1}{2}\right) \implies \sin(x-1) = f(-x/2) \implies f(x) = \sin(-2x-1)$$

Therefore the solution of the PDE is

$$u(x,t) = f\left(x-\frac{y}{2}\right) = \sin\left[-2\left(x-\frac{y}{2}\right)-1\right] = \sin(y-2x-1)$$

Problem 7:

Example 7

Find the solution surface for the linear PDE

$$3\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} = 10$$

that contains the lines $y = 2x$, $u = 2x/5$. Show that the projection of the initial curve in the (x,y) plane is nowhere tangent to a base characteristic curve.

Solution The Lagrange–Charpit equations (8) for the PDE are

$$\frac{dx}{3} = \frac{dy}{4} = \frac{du}{10}$$

From this, we have

$$\frac{dx}{dy} = \frac{3}{4}, \qquad \frac{du}{dy} = \frac{10}{4}$$

Integration of these gives characteristic curves

$$x = \frac{3}{4}y + A, \qquad u = \frac{5}{2}y + B$$

where A and B are arbitrary constants. From the first of the above equation, we have

$$x - \frac{3}{4}y = A$$

Specifying B as a function of A gives a solution surface,

$$u = \frac{5}{2}y + f\left(x - \frac{3}{4}y\right)$$

where $f(\cdot)$ is an arbitrary function. For solution curves to pass through the initial data, $y = 2x$, $u = 2x/5$, we set

$$\frac{2}{5}x = 5x + f\left(x - \frac{3}{4}2x\right) \quad \Longrightarrow \quad -\frac{23}{5}x = f\left(-\frac{x}{2}\right) \quad \Longrightarrow \quad f(x) = \frac{46}{5}x$$

Therefore the solution of the PDE is

$$u(x,t) = \frac{5}{2}y + f\left(x - \frac{3}{4}y\right) = \frac{5}{2}y + \frac{46}{5}\left(x - \frac{3}{4}y\right) = \frac{46}{5}x - \frac{22}{5}y$$

The solution surface is a plane defined for all x and y. Base characteristic curves are straight lines $y = 4x/3 + A$ with slope $4/3$. Since the projection of the initial curve in the (x,y) plane is the line $y = 2x$ with slope 2, it is nowhere tangent to a base characteristic curve.

Problem 8:

Example 5

Find the solution of the Cauchy problem governed by the semilinear PDE

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xe^{-u}$$

subject to the Cauchy data $u = 0$ on $y = x^2$.

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Solution The given PDE is of the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c$$

where

$$a = x, \qquad b = y, \qquad c = xe^{-u}$$

From the Lagrange–Charpit equations (8), we have

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{xe^{-u}}$$

Using the Lagrange–Charpit equation, we have the reciprocal of the slope of characteristic curves

$$\frac{dy}{dx} = \frac{y}{x}$$

Separating the variables and integrating to obtain

$$y = Ax$$

where A is an arbitrary constant. Further, we have

$$\frac{du}{dx} = e^{-u}$$

which leads to

$$e^{u} = x + B$$

where B is an arbitrary constant. Thus,

$$\frac{y}{x} = A, \qquad e^{u} - x = B$$

is a two-parameter family of characteristic curves. The general solution is expressed by writing $B = f(A)$ as follows

Thus,

$$\frac{y}{x} = A, \qquad e^u - x = B$$

is a two-parameter family of characteristic curves. The general solution is expressed by writing $B = f(A)$ as follows

$$e^u - x = f\left(\frac{y}{x}\right)$$

where $f(\cdot)$ is an arbitrary function. Applying the Cauchy data $u = 0$ on $y = x^2$

$$1 - x = f(x) \qquad \Longrightarrow \qquad f\left(\frac{y}{x}\right) = 1 - \frac{y}{x}$$

Therefore the solution of the PDE is

$$e^u - x = 1 - \frac{y}{x}$$

or

$$u(x,y) = \ln\left(x + 1 - \frac{y}{x}\right)$$