My Inspiration
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and
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Subject Teacher
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Practical No 5 : Find complete integral using Jacobi's Method

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Problem 1:

Solve the PDE ${p_1}^3 + {p_2}^2 + p_3 = 1$ by Jacobi's method

Solution

Let

$$f = p_1^3 + {p_2}^2 + p_3 - 1 = 0 \tag{1}$$

$$\frac{\partial f}{\partial x_1} = f_{x_1} = 0$$
$$\frac{\partial f}{\partial x_2} = f_{x_2} = 0$$

Solution of Problem 1 Continue...

$$\frac{\partial f}{\partial x_3} = f_{x_3} = 0$$

$$\frac{\partial f}{\partial p_1} = f_{p_1} = 3{p_1}^2$$

$$\frac{\partial f}{\partial p_2} = f_{p_2} = 2p_2$$

$$\frac{\partial f}{\partial p_3} = f_{p_3} = -1$$

Jacobi's Auxiliary equation is,

$$\frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}}$$

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Solution of Problem 1 Continue...

$$\frac{dx_1}{-3{p_1}^2} = \frac{dx_2}{2p_2} = \frac{dx_3}{-1} = \frac{dp_1}{0} = \frac{dp_2}{0} = \frac{dp_3}{0}$$

$$EachRatio = \frac{dp_1}{0} \implies \mathrm{dp}_1 = 0$$

Integrating,

$p_1 = a$.....where a is constant

Similarly, $EachRatio = \frac{dp_2}{0} \implies \mathrm{dp}_2 = 0$

Integrating,

$p_2 = b$.....where b is constant

Using $p_1 = a$ and $p_2 = b$ in equation (1)

$$a^3 + b^2 + p_3 = 1 \implies p_3 = 1 - a^3 - b^2$$

Solution of Problem 1 Continue...

Using the values of p_1, p_2 and p_3 in

$$p_1 \; dx_1 + p_2 \; dx_2 + p_3 \; dx_3 = dz$$

Consider, $p_1 \; dx_1 + p_2 \; dx_2 + p_3 \; dx_3 = dz$

$$\therefore \; a \quad dx_1 + b \; dx_2 + (1 - a^3 - b^2) \; dx_3 = dz$$

Integrating we get,

$$a \; x_1 + b \; x_2 + (1 - a^3 - b^2) \; x_3 = z + c$$

Required Solution.

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Problem 2:

Solve the PDE $2\,p_1\,x_1\,x_3 + 3\,p_2\,{x_3}^2 + {p_2}^2\,p_3 = 0$ by Jacobi's method

Solution

Let

$$f = 2\,p_1\,x_1\,x_3 + 3\,p_2\,{x_3}^2 + {p_2}^2\,p_3 = 0 \tag{2}$$

$$\frac{\partial f}{\partial x_1} = f_{x_1} = 2\,p_1\,x_3$$

$$\frac{\partial f}{\partial x_2} = f_{x_2} = 0$$

Solution of Problem 2 Continue...

$$\frac{\partial f}{\partial x_3} = f_{x_3} = 2\,p_1\,x_1 + 6\,p_2\,x_3$$

$$\frac{\partial f}{\partial p_1} = f_{p_1} = 2\,x_1\,x_3$$

$$\frac{\partial f}{\partial p_2} = f_{p_2} = 3\,{x_3}^2 + 2\,p_2\,p_3$$

$$\frac{\partial f}{\partial p_3} = f_{p_3} = {p_2}^2$$

Jacobi's Auxiliary equation is,

$$\frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}}$$

Solution of Problem 2 Continue...

$$\frac{dx_1}{-(2\ x_1\ x_3)} = \frac{dx_2}{-(3\ {x_3}^2 + 2\ p_2\ p_3)} = \frac{dx_3}{-{p_2}^2} = \frac{dp_1}{2\ p_1\ x_3} = \frac{dp_2}{0} = \frac{dp_3}{2\ p_1\ x_1 + 6\ p_2\ x_3}$$

$$EachRatio = \frac{dp_2}{0} \implies \mathrm{dp}_2 = 0$$

Integrating,

$p_2 = a$.....where a is constant

Similarly, Consider,

$$\frac{dx_1}{-(2\,x_1\,x_3)} = \frac{dp_1}{2\,p_1\,x_3}$$

$$\frac{dx_1}{-x_1} = \frac{dp_1}{p_1}$$

Integrating,

$$-\ln x_1 = \ln p_1 - \ln b$$

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Solution of Problem 2 Continue...

$\ln x_1 + \ln p_1 = \ln b$

$\ln x_1 p_1 = \ln b \implies x_1 p_1 = b$

$p_1 = \dfrac{b}{x_1}$.....where b is constant

Using $p_1 = \dfrac{b}{x_1}$ and $p_2 = a$ in equation (2)

$2\,\dfrac{b}{x_1}\,x_1\,x_3 + 3\,a\,{x_3}^2 + a^2\,p_3 = 0$

$a^2\,p_3 = -2\,b\,x_3 - 3\,a\,{x_3}^2$

$p_3 = -\dfrac{1}{a^2}(2\,b\,x_3 + 3\,a\,{x_3}^2)$

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Solution of Problem 2 Continue...

Using the values of p_1, p_2 and p_3 in

$p_1\ dx_1 + p_2\ dx_2 + p_3\ dx_3 = dz$

Consider,

$$p_1\ dx_1 + p_2\ dx_2 + p_3\ dx_3 = dz$$

$$\therefore \quad \frac{b}{x_1}\ dx_1 + a\ dx_2 - \frac{1}{a^2}(2\,b\,x_3 + 3\,a\,{x_3}^2)dx_3 = dz$$

Integrating we get,

$$b\ \ln x_1 + a\,x_2 - \frac{1}{a^2}\left(2\,b\frac{{x_3}^2}{2} + 3\,a\frac{{x_3}^3}{3}\right) = z + c$$

$$b\ \ln x_1 + a\,x_2 - \frac{b}{a^2}\,{x_3}^2 - \frac{{x_3}^3}{a} = z + c$$

Required Solution.

Problem 3:

Solve the PDE $p_1\, x_1 + p_2\, x_2 = {p_3}^2$ by Jacobi's method

Solution

Let

$$f = p_1\, x_1 + p_2\, x_2 - {p_3}^2 = 0 \tag{3}$$

$$\frac{\partial f}{\partial x_1} = f_{x_1} = p_1$$

$$\frac{\partial f}{\partial x_2} = f_{x_2} = p_2$$

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Solution of Problem 3 Continue...

$$\frac{\partial f}{\partial x_3} = f_{x_3} = 0$$

$$\frac{\partial f}{\partial p_1} = f_{p_1} = x_1$$

$$\frac{\partial f}{\partial p_2} = f_{p_2} = x_2$$

$$\frac{\partial f}{\partial p_3} = f_{p_3} = -2p_3$$

Jacobi's Auxiliary equation is,

$$\frac{dx_1}{-f_{p_1}} = \frac{dx_2}{-f_{p_2}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_1}{f_{x_1}} = \frac{dp_2}{f_{x_2}} = \frac{dp_3}{f_{x_3}}$$

Solution of Problem 3 Continue...

$$\frac{dx_1}{-x_1} = \frac{dx_2}{-x_2} = \frac{dx_3}{-(-2p_3)} = \frac{dp_1}{p_1} = \frac{dp_2}{p_2} = \frac{dp_3}{0}$$

$$EachRatio = \frac{dp_3}{0} \implies \mathrm{dp}_3 = 0$$

Integrating,

$p_3 = a$.....where a is constant

Similarly, Consider,

$$\frac{dx_1}{-x_1} = \frac{dp_1}{p_1}$$

Integrating,

$$-\ln x_1 = \ln p_1 - \ln b$$

Solution of Problem 3 Continue...

$\ln x_1 + \ln p_1 = \ln b$

$\ln x_1 p_1 = \ln b \implies x_1 p_1 = b$

$p_1 = \frac{b}{x_1}$.....where b is constant

Using $p_1 = \frac{b}{x_1}$ and $p_3 = a$ in equation (3)

$2\frac{b}{x_1} x_1 + p_2 x_2 - a^2 = 0$

$p_2 x_2 = a^2 - b$

$p_2 = \frac{a^2 - b}{x_2}$

Solution of Problem 3 Continue...

Using the values of p_1, p_2 and p_3 in

$p_1 \; dx_1 + p_2 \; dx_2 + p_3 \; dx_3 = dz$

Consider,

$p_1 \; dx_1 + p_2 \; dx_2 + p_3 \; dx_3 = dz$

$$\therefore \quad \frac{b}{x_1} \; dx_1 + \frac{a^2 - b}{x_2} \; dx_2 + adx_3 = dz$$

Integrating we get,

$b \ln x_1 + (a^2 - b) \ln x_2 + a \, x_3 = z + c$

$b \ln x_1 + (a^2 - b) \ln x_2 + a \, x_3 = z + c$

Required Solution.

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Problem 4:

Solve the PDE $p_1\, p_2\, p_3 = z^3\, x_1\, x_2\, x_3$ by Jacobi's method

Solution

Let

$$p_1\, p_2\, p_3 = z^3\, x_1\, x_2\, x_3$$

Dividing by z^3

$$\left(\frac{1}{z}\, p_1\right)\left(\frac{1}{z}\, p_2\right)\left(\frac{1}{z}\, p_3\right) = x_1\, x_2\, x_3$$

Put $u = \log z$

Differentiate w.r.t. x_1, we get

$$\therefore \quad \frac{\partial u}{\partial x_1} = \frac{1}{z}\frac{\partial z}{\partial x_1} = \frac{1}{z}\, p_1$$

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Solution of Problem 4 Continue...

$$\text{Take } P_1 = \frac{1}{z}\,p_1, \quad P_2 = \frac{1}{z}\,p_2, \quad P_3 = \frac{1}{z}\,p_3$$

Equation becomes,

$$P_1\,P_2\,P_3 = x_1\,x_2\,x_3$$

$$\mathrm{f} = \mathrm{P}_1\,P_2\,P_3 - x_1\,x_2\,x_3 = 0 \tag{4}$$

$$\frac{\partial f}{\partial x_1} = f_{x_1} = -x_2\,x_3,$$
$$\frac{\partial f}{\partial x_2} = f_{x_2} = -x_1\,x_3,$$
$$\frac{\partial f}{\partial x_3} = f_{x_3} = -x_1\,x_2;$$

Solution of Problem 4 Continue...

$$\frac{\partial f}{\partial P_1} = f_{P_1} = P_2 P_3$$
$$\frac{\partial f}{\partial P_2} = f_{P_2} = P_1 P_3$$
$$\frac{\partial f}{\partial P_3} = f_{P_3} = P_1 P_2$$

Jacobi's Auxiliary equation is,

$$\frac{dx_1}{-f_{P_1}} = \frac{dx_2}{-f_{P_2}} = \frac{dx_3}{-f_{P_3}} = \frac{dP_1}{f_{x_1}} = \frac{dP_2}{f_{x_2}} = \frac{dP_3}{f_{x_3}}$$

$$\frac{dx_1}{-P_2 P_3} = \frac{dx_2}{-P_1 P_3} = \frac{dx_3}{-P_1 P_2} = \frac{dP_1}{-x_2 x_3} = \frac{dP_2}{-x_1 x_3} = \frac{dP_3}{-x_1 x_2}$$

Solution of Problem 4 Continue...

Consider

$$\frac{dx_1}{-P_2P_3} = \frac{dP_1}{-x_2x_3}$$

$$\therefore \frac{P_1dx_1}{P_1P_2P_3} = \frac{dP_1}{x_2x_3}$$

$$\therefore \frac{P_1dx_1}{x_1x_2x_3} = \frac{dP_1}{x_2x_3} \text{ ..... By Equation (49)}$$

$$\therefore \frac{dx_1}{x_1} = \frac{dP_1}{P_1}$$

Integrating,

$$\therefore \log x_1 = \log P_1 - \log a$$

$$\therefore \log x_1 + \log a = \log P_1$$

$P_1 = ax_1$.....where a is constant

Solution of Problem 4 Continue...

Similarly, Consider,

$$\frac{dx_2}{-P_1P_3} = \frac{dP_2}{-x_1x_3}$$

$$\therefore \quad \frac{P_2\,dx_2}{P_1P_2P_3} = \frac{dP_2}{x_1x_3}$$

$$\therefore \quad \frac{P_2\,dx_2}{x_1x_2x_3} = \frac{dP_2}{x_1x_3} \text{... by equation (49)}$$

$$\therefore \quad \frac{dx_2}{x_2} = \frac{dP_2}{P_2}$$

Integrating,

$$\therefore \quad \log x_2 = \log P_2 - \log b$$

$$\therefore \quad \log x_2 + \log b = \log P_2$$

$$P_2 = bx_2 \text{.....where b is constant}$$

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Solution of Problem 4 Continue...

Using $P_1 = ax_1$ and $P_2 = bx_2$ in equation (4)

$$\therefore \text{a} \quad x_1\, b\, x_2\, P_3 - x_1x_2x_3 = 0$$

$$\therefore \text{a} \quad x_1\, b\, x_2\, P_3 = x_1x_2x_3$$

$$\therefore \text{P}_3 = \frac{x_3}{ab}$$

$$P_3 = \frac{x_3}{ab}$$

Using the values of p_1, p_2 and p_3 in in

$$P_1 \;\; dx_1 + P_2 \;\; dx_2 + P_3 \;\; dx_3 = dz$$

$$\therefore \quad \text{a } x_1 \;\; dx_1 + bx_2 \;\; dx_2 + \frac{x_3}{ab}dx_3 = dz$$

Integrating we get,

$$\frac{ax_1{}^2}{2} + \frac{bx_2{}^2}{2} + \frac{x_3{}^2}{2ab}z + c$$

$a^2bx_1{}^2 + ab^2x_2{}^2 + x_3{}^2 = 2abz + c$... Required Solution.

Practical No 5 : Find complete integral using Jacobi's Method

Problem 5:

Solve the PDE $p^2 x + q^2 y = z$ by Jacobi's method

Solution

Jacobi's method is used for solving first order partial differential equation involving 3 or more independent variables. Here x and y are independent and z is dependent variable. So we consider z as independent variable

if and only if $u(x, y, z) = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0 \implies \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p = 0$$

Practical No 5 : Find complete integral using Jacobi's Method

Solution of Problem 5 Continue...

Let

$$u_1 = \frac{\partial u}{\partial x}, \quad u_2 = \frac{\partial u}{\partial y}, \quad u_3 = \frac{\partial u}{\partial z}$$

$$\therefore \ u_1 + u_3 \ p = 0$$

$$\therefore \ p = - \ \frac{u_1}{u_3}$$

Similarly, $q = -\frac{u_2}{u_3}$

Using this in (1)

$$\frac{{u_1}^2}{{u_3}^2} x + \frac{{u_2}^2}{{u_3}^2} y = z$$

$$\therefore \ u_1^2 x + {u_2}^2 y = {u_3}^2 z$$

Solution of Problem 5 Continue...

Let

$$f = u_1^2\, x + {u_2}^2\, y - {u_3}^2\, z = 0 \tag{5}$$

$$\frac{\partial f}{\partial x} = f_x = {u_1}^2,$$
$$\frac{\partial f}{\partial y} = f_y = {u_2}^2,$$
$$\frac{\partial f}{\partial z} = f_z = -{u_3}^2;$$
$$\frac{\partial f}{\partial u_1} = f_{u_1} = 2u_1\, x$$
$$\frac{\partial f}{\partial u_2} = f_{u_2} = 2u_2\, y \text{ and } \frac{\partial f}{\partial u_3} = f_{u_3} = -2u_3\, z$$

Solution of Problem 5 Continue...

Jacobi's Auxiliary equation is,

$$\frac{dx}{-f_{u_1}} = \frac{dy}{-f_{u_2}} = \frac{dz}{-f_{u_3}} = \frac{du_1}{f_x} = \frac{du_2}{f_y} = \frac{du_3}{f_z}$$

$$\frac{dx}{-2u_1\,x} = \frac{dy}{-2u_2\,y} = \frac{dz}{-2u_3\,z} = \frac{du_1}{{u_1}^2} = \frac{du_2}{{u_2}^2} = \frac{du_3}{-{u_3}^2}$$

Consider

$$\frac{dx}{-2u_1\,x} = \frac{du_1}{{u_1}^2}$$
$$\frac{dx}{-2\,x} = \frac{du_1}{u_1}$$

Solution of Problem 5 Continue...

Integrating,

$$\therefore \quad -\frac{1}{2}\log x = \log u_1 - \log a$$

$$\therefore \quad \log {u_1}^2 + \log x = \log a$$

$$\therefore \quad \log {u_1}^2 x = \log a$$

$$\therefore \quad {u_1}^2 x = a$$

$u_1 = \sqrt{\frac{a}{x}}$.....where a is constant

Similarly, Consider,

$$\frac{dy}{-2u_2\,y} = \frac{du_2}{{u_2}^2}$$

$$\frac{dy}{-2\,y} = \frac{du_2}{u_2}$$

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Solution of Problem 5 Continue...

Integrating,

$$\therefore \quad -\frac{1}{2}\log y = \log u_2 - \log b$$

$$\therefore \quad \log {u_2}^2 + \log y = \log b$$

$$\therefore \quad \log {u_2}^2 y = \log b \therefore \quad {u_2}^2 y = b$$

$$u_2 = \sqrt{\frac{b}{y}}$$ .....where b is constant

Using $u_1 = \sqrt{\frac{a}{x}}$ and $u_2 = \sqrt{\frac{b}{y}}$ in equation (5)

$$\therefore \quad \frac{a}{x}x + \frac{b}{y}y - {u_3}^2 z = 0 \implies \therefore u_3^2 z = a + b$$

$$u_3 = \sqrt{\frac{a+b}{z}}$$

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Solution of Problem 5 Continue...

Using the values of u_1, u_2 and u_3 in in

$$u_1 \; dx + u_2 \; dy + u_3 \; dz = du$$

$$\therefore \; \sqrt{\frac{a}{x}} \; dx + \sqrt{\frac{b}{y}} \; dy + \sqrt{\frac{a+b}{z}} \; dz = du$$

$$\therefore \; \sqrt{a} \; \frac{dx}{\sqrt{x}} + \sqrt{b} \; \frac{dy}{\sqrt{y}} + \sqrt{a+b} \; \frac{dz}{\sqrt{z}} = du$$

Integrating we get,

$$2\sqrt{ax} + 2\sqrt{by} + 2\sqrt{(a+b)z} = u + c$$

But $u(x, y, z) = 0$

$$\sqrt{ax} + \sqrt{by} + \sqrt{(a+b)z} = c$$

Required Solution.