My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Practical No 4 : Find complete integral of first order PDE using Charpit's Method.

Subject Teacher
Santosh Dhamone

Assistant Professor in Mathematics
Art's Commerce and Science College,Onde
Tal:- Vikramgad, Dist:- Palghar

ssdhamone@acscollegeonde.ac.in

July 29, 2023

Contents

Practical No 4 : Find complete integral of first order PDE using Charpit's Method

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Problem 1:

Solve the PDE $(p^2 + q^2)y = qz$ by Charpit's method

Solution

Let

$$f = (\,p^2 + q^2)y - qz = 0 \tag{1}$$

$$\frac{\partial f}{\partial x} = f_x = 0$$

$$\frac{\partial f}{\partial y} = f_y = p^2 + q^2$$

Practical No 4 : Find complete integral of first order PDE using Charpit's Method

Solution of Problem 1 Continue...

$$\frac{\partial f}{\partial z} = f_z = -q$$

$$\frac{\partial f}{\partial p} = f_p = 2py$$

$$\frac{\partial f}{\partial q} = f_q = 2qy - z$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{2p^2y + 2q^2y - qz} = \frac{-dp}{0 + (-pq)} = \frac{-dq}{p^2 + q^2 - q^2}$$

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Solution of Problem 1 Continue...

$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{2p^2y + 2q^2y - qz} = \frac{-dp}{-pq} = \frac{-dq}{p^2}$$

Consider two last ratios

$$\frac{dp}{pq} = \frac{-dq}{p^2} \implies \frac{dp}{q} = \frac{-dq}{p} \implies \text{pdp} = \text{-qdq}$$

Integrating,

$$\frac{p^2}{2} + \frac{q^2}{2} = a \implies \text{p}^2 + q^2 = a$$

Using $p^2 + q^2 = a$ in equation (1)

$$ay = qz \implies q = \frac{ay}{z}$$

Using $q = \frac{ay}{z}$ in $p^2 + q^2 = a$

Solution of Problem 1 Continue...

$$p^2 + \left(\frac{ay}{z}\right)^2 = a \implies p^2 = a - \frac{a^2y^2}{z^2}$$

$$\implies p^2 = \frac{az^2 - a^2y^2}{z^2} \implies p = \frac{\sqrt{az^2 - a^2y^2}}{z}$$

Consider, $p\ dx + q\ dy = dz$

$$\therefore \frac{\sqrt{az^2 - a^2y^2}}{z}\ dx + \frac{ay}{z}dy = dz$$

$$\sqrt{az^2 - a^2y^2}\ dx + ay\ dy = z\ dz$$

$$\sqrt{a}(\sqrt{z^2 - ay^2})\ dx = z\ dz - ay\ dy$$

$$\sqrt{a}\ dx = \frac{zdz - aydy}{\sqrt{z^2 - ay^2}}$$

Integrating we get,

$\sqrt{a}\ x = \sqrt{z^2 - ay^2} + b$ ...Required Solution.

Practical No 4 : Find complete integral of first order PDE using Charpit's Method

Problem 2:

Solve the PDE $p = (z + qy)^2$ by Charpit's method

Solution

Let

$$f = p - (z + qy)^2 = 0 \tag{2}$$

$$\frac{\partial f}{\partial x} = f_x = 0$$

$$\frac{\partial f}{\partial y} = f_y = -2(z + qy).q$$

Solution of Problem 2 Continue...

$$\frac{\partial f}{\partial z} = f_z = -2(z + qy)$$

$$\frac{\partial f}{\partial p} = f_p = 1$$

$$\frac{\partial f}{\partial q} = f_q = -2(z + qy).y$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{1} = \frac{dy}{-2y(z + qy)} = \frac{dz}{p - 2yq(z + qy)} =$$

$$\frac{-dp}{-2p(z + qy)} = \frac{-dq}{-2q(z + qy) - 2q(z + qy)}$$

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Solution of Problem 2 Continue...

$$\text{Consider } \frac{dy}{-2y(z+qy)} = \frac{-dp}{-2p(z+qy)}$$

$$\therefore \frac{dy}{-y} = \frac{dp}{p}$$

Integrating,

$$-\ln y = \ln p - \ln a \implies \ln y + \ln p = \ln a \implies \text{yp} = \text{a}$$

$$\therefore p = \frac{a}{y}$$

$$\text{Using } p = \frac{a}{y} \text{ in equation (2)}$$

$$\frac{a}{y} = (z+qy)^2 \implies \text{z} + \text{qy} = \sqrt{\frac{a}{y}} \implies \text{qy} = \sqrt{\frac{a}{y}} - z$$

Solution of Problem 2 Continue...

$$\therefore q = \frac{1}{y}\left(\sqrt{\frac{a}{y}} - z\right)$$

Consider, $p\ dx + q\ dy = dz$

$$\therefore \frac{a}{y}\ dx + \ \frac{1}{y}\left(\sqrt{\frac{a}{y}} - z\right) dy = dz$$

$$adx + \sqrt{\frac{a}{y}}dy = ydz + zdy$$

$$a\ dx + \sqrt{\frac{a}{y}}dy = d(yz)$$

Integrating we get,

$$a\,x + 2\sqrt{ay} = yz + b$$

$$ax + 2\sqrt{ay} - yz = b$$

This is Required Complete Integral.

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Problem 3:

Solve the PDE $2xz - px^2 - 2qxy + pq = 0$ by Charpit's method

Solution

Let

$$f = 2xz - px^2 - 2qxy + pq = 0 \tag{3}$$

$$\frac{\partial f}{\partial x} = f_x = 2z - 2px - 2qy$$

$$\frac{\partial f}{\partial y} = f_y = -2qx$$

Practical No 4 : Find complete integral of first order PDE using Charpit's Method

Solution of Problem 3 Continue...

$$\frac{\partial f}{\partial z} = f_z = 2x$$

$$\frac{\partial f}{\partial p} = f_p = x^2 + q$$

$$\frac{\partial f}{\partial q} = f_q = p - 2xy$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{x^2 + q} = \frac{dy}{p - 2xy} = \frac{dz}{p(x^2 + q) + q(p - 2xy)} = \frac{-dp}{2(z - px - qy) + 2px} = \frac{-dq}{-2qx + 2qx}$$

$$\frac{dx}{x^2 + q} = \frac{dy}{p - 2xy} = \frac{dz}{px^2 + 2pq - 2qxy} = \frac{-dp}{2z - 2qy} = \frac{dq}{0}$$

Solution of Problem 3 Continue...

Consider,

$$\text{Each Ratio} = \frac{dq}{0}$$

$$\therefore \text{dq} = 0$$

Integrating,we get

$$\therefore \quad q = a$$

Using $q = a$ in equation (3)

$$2xz - px^2 - 2axy + pa = 0$$

$$\therefore \text{p(a - x}^2) = 2axy - 2xz$$

$$\therefore \quad \text{p} = \frac{2x(ay - z)}{a - x^2}$$

Solution of Problem 3 Continue...

Consider, $p\ dx + q\ dy = dz$

$$\therefore \frac{2x(ay-z)}{a-x^2}\ dx + \text{a}\ \ \text{dy} = \text{dz}$$

Divide throughout by $(ay-z)$, we get,

$$\therefore \frac{2x}{a-x^2}\ dx + \frac{a}{(ay-z)}\ dy = \frac{dz}{(ay-z)}$$

$$\therefore \text{-} \frac{-2x}{a-x^2}\ dx + \frac{ady-dz}{(ay-z)} = 0$$

Integrating we get,

$$-\ln(a-x^2) + \ln(ay-z) = \ln b$$

$$\ln\frac{ay-z}{a-x^2} = \ln b$$

$\dfrac{ay-z}{a-x^2} = b$.... This is Required Complete Integral.

Practical No 4 : Find complete integral of first order PDE using Charpit's Method

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Problem 4:

Solve the PDE $2(z + xp + yq) = yp^2$ by Charpit's method

Solution

Let

$$f = 2(z + xp + yq) - yp^2 = 0 \tag{4}$$

$$\frac{\partial f}{\partial x} = f_x = 2p$$

$$\frac{\partial f}{\partial y} = f_y = 2q - p^2$$

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Solution of Problem 4 Continue...

$$\frac{\partial f}{\partial z} = f_z = 2$$

$$\frac{\partial f}{\partial p} = f_p = 2x - 2yp$$

$$\frac{\partial f}{\partial q} = f_q = 2y$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{2x - 2yp} = \frac{dy}{2y} = \frac{dz}{2xp - 2yp^2 + 2yq} = \frac{-dp}{2p + 2p} = \frac{-dq}{2q - p^2 + 2q}$$

$$\frac{dx}{2x - 2yp} = \frac{dy}{2y} = \frac{dz}{2xp - 2yp^2 + 2yq} = \frac{-dp}{4p} = \frac{-dq}{4q - p^2}$$

Solution of Problem 4 Continue...

Consider,

$$\frac{dy}{2y} = \frac{-dp}{4p}$$

Integrating,we get

$$2\ln y = -\ln p + \ln a$$

$$\ln y^2 + \ln p = \ln a \implies \ln y^2 p = \ln a$$

$$\therefore \quad p = \frac{a}{y^2}$$

Using $p = \frac{a}{y^2}$ in equation (4)

$$2z + 2x\frac{a}{y^2} + 2yq - y\left(\frac{a}{y^2}\right)^2 = 0$$

$$\therefore \quad 2z + \frac{2ax}{y^2} + 2yq - \left(\frac{a^2}{y^3}\right) = 0$$

Solution of Problem 4 Continue...

$$\therefore \quad q = \frac{a^2}{2y^4} - \frac{z}{y} - \frac{xa}{y^3}$$

Consider, $p\ dx + q\ dy = dz$

$$\therefore \quad \frac{a}{y^2}\ dx + \left(\frac{a^2}{2y^4} - \frac{z}{y} - \frac{xa}{y^3}\right)\ dy = dz$$

$$\therefore \quad \frac{a}{y^2}\ dx + \ \frac{a^2}{2y^4}dy - \frac{z}{y}dy - \frac{xa}{y^3}dy = dz$$

$$a\left(\frac{ydx - xdy}{y^3}\right) + \frac{a^2}{2}\frac{dy}{y^3} = ydz + zdy$$

$$ad(\frac{x}{y}) + \frac{a^2}{2}\frac{dy}{y^3} = d(yz)$$

Integrating we get,

Solution of Problem 4 Continue...

Integrating we get,

$$a\frac{x}{y} - \frac{a^2}{4y^2} = yz + b$$

$$\frac{ax}{y} - \frac{a^2}{4y^2} - yz = b$$

This is Required Complete Integral.

Problem 5:

Solve the PDE $pxy + pq + qy = yz$ by Charpit's method

Solution

Let

$$f = pxy + pq + qy - yz = 0 \tag{5}$$

$$\frac{\partial f}{\partial x} = f_x = py$$

$$\frac{\partial f}{\partial y} = f_y = q - z$$

Solution of Problem 5 Continue...

$$\frac{\partial f}{\partial z} = f_z = -y$$

$$\frac{\partial f}{\partial p} = f_p = xy + q$$

$$\frac{\partial f}{\partial q} = f_q = p + y$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{xy+q} = \frac{dy}{p+y} = \frac{dz}{xyp+qp+pq+yq} = \frac{-dp}{py-yp} = \frac{-dq}{q-z-qy}$$

$$\frac{dx}{xy+q} = \frac{dy}{p+y} = \frac{dz}{xyp+2pq+yq} = \frac{-dp}{0} = \frac{-dq}{q-z-qy}$$

Solution of Problem 5 Continue...

Consider,

$$EachRatio = \frac{dp}{0}$$

$$\therefore \text{dp} = 0$$

Integrating,we get

$$\therefore \quad p = a$$

Using $p = a$ in equation (5)

$$axy + aq + qy - yz = 0$$

$$q(a + y) = yz - axy$$

$$\therefore \quad q = \frac{y(z - ax)}{a + y}$$

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Solution of Problem 5 Continue...

Consider, $p \ dx + q \ dy = dz$

$$\therefore \quad a \quad dx + \frac{y(z-ax)}{a+y} dy = dz$$

$$\therefore \quad \frac{y(z-ax)}{a+y} dy = dz - a \ dx$$

$$\therefore \quad \frac{y}{a+y} dy = \frac{dz-adx}{(z-ax)}$$

$$\therefore \quad \frac{y+a-a}{a+y} dy = \frac{dz-adx}{(z-ax)}$$

$$\therefore \quad \left(1-\frac{a}{a+y}\right) dy = \frac{dz-adx}{(z-ax)}$$

Solution of Problem 5 Continue...

Integrating we get,

$$y - a\ln(a+y) = \ln(z-ax) + b$$

$$y - \ln(a+y)^a - \ln(z-ax) = b$$

$$y - [\ln(a+y)^a + \ln(z-ax)] = b$$

$$y - \ln(a+y)^a(z-ax) = b$$

$$y - b = \ln(a+y)^a(z-ax)$$

$$(a+y)^a(z-ax) = e^{y-b}$$

$$(a+y)^a(z-ax) = ce^y$$

This is Required Complete Integral.

Problem 6:

Solve the PDE $p^2x + q^2y = z$ by Charpit's method

Solution

Let

$$f = p^2x + q^2y - z = 0 \tag{6}$$

$$\frac{\partial f}{\partial x} = f_x = p^2$$
$$\frac{\partial f}{\partial y} = f_y = q^2$$

Solution of Problem 6 Continue...

$$\frac{\partial f}{\partial z} = f_z = -1$$

$$\frac{\partial f}{\partial p} = f_p = 2px$$

$$\frac{\partial f}{\partial q} = f_q = 2qy$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2p^2x + 2q^2y} = \frac{-dp}{p^2 - p} = \frac{-dq}{q^2 - q}$$

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Solution of Problem 6 Continue...

Consider,

$$\text{Each Ratio} = \frac{p^2dx + 2pxdp}{2p^2x}$$

Similarly,

$$\text{Each Ratio} = \frac{q^2dy + 2qydq}{2q^2y}$$

$$\frac{p^2dx + 2pxdp}{2p^2x} = \frac{q^2dy + 2qydq}{2q^2y}$$

Integrating,we get

$$\ln p^2x = \ln q^2y + \ln a$$

$$\ln p^2x = \ln aq^2y \implies \mathrm{p}^2x = aq^2y \implies \mathrm{p}^2 = \frac{ayq^2}{x}$$

$$\therefore \quad p = \sqrt{\frac{ay}{x}}q$$

Solution of Problem 6 Continue...

Using $p = \sqrt{\frac{ay}{x}}q$ in equation (6)

$$\frac{ayq^2}{x}x + q^2y = z$$

$$q^2y(1+a) = z \implies q^2 = \frac{z}{y(1+a)}$$

$$\therefore \quad q = \sqrt{\frac{z}{y(1+a)}}$$

using value of q in p

$$p = \sqrt{\frac{ay}{x}}\sqrt{\frac{z}{y(1+a)}}$$

$$\therefore \quad p = \sqrt{\frac{az}{x(1+a)}}$$

Solution of Problem 6 Continue...

Consider, $p\ dx + q\ dy = dz$

$$\sqrt{\frac{az}{x(1+a)}}dx + \sqrt{\frac{z}{y(1+a)}}dy = dz$$

$$\sqrt{\frac{a}{(1+a)}}\frac{dx}{\sqrt{x}} + \sqrt{\frac{1}{(1+a)}}\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

Integrating we get,

$$\sqrt{\frac{a}{(1+a)}}2\sqrt{x} + \sqrt{\frac{1}{(1+a)}}2\sqrt{y} = 2\sqrt{z} + b$$

$$\sqrt{ax} + \sqrt{y} = \sqrt{a+1}(\sqrt{z} + b)$$

This is Required Complete Integral.

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Problem 7:

Solve the PDE $p^2x + qy = z$ by Charpit's method

Solution

Let

$$f = p^2x + qy - z = 0 \tag{7}$$

$$\frac{\partial f}{\partial x} = f_x = p^2$$

$$\frac{\partial f}{\partial y} = f_y = q$$

Solution of Problem 6 Continue...

$$\frac{\partial f}{\partial z} = f_z = -1$$

$$\frac{\partial f}{\partial p} = f_p = 2px$$

$$\frac{\partial f}{\partial q} = f_q = y$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{2px} = \frac{dy}{y} = \frac{dz}{2p^2x + qy} = \frac{-dp}{p^2 - p} = \frac{-dq}{q - q}$$

Solution of Problem 7 Continue...

Consider,

$$EachRatio = \frac{dq}{0}$$

$$\therefore \text{dq} = 0$$

Integrating,we get

$$\therefore \quad q = a$$

Using $q = a$ in equation (7)

$$p^2x + ay - z = 0 \implies \text{p}^2x = z - ay \implies \text{p}^2 = \frac{z - ay}{x}$$

$$p = \sqrt{\frac{z - ay}{x}}$$

$$\therefore \quad p = \sqrt{\frac{z - ay}{x}}$$

Practical No 4 : Find complete integral of first order PDE using Charpit's Method

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Solution of Problem 7 Continue...

Consider, $p \; dx + q \; dy = dz$

$$\sqrt{\frac{z-ay}{x}}dx + ady = dz$$

$$\sqrt{z-ay}\frac{dx}{\sqrt{x}} = dz - ady$$

$$\frac{dx}{\sqrt{x}} = \frac{dz-ady}{\sqrt{z-ay}}$$

Integrating we get,

$$2\sqrt{x} = 2\sqrt{z-ay} + b$$

$$\sqrt{x} - \sqrt{z-ay} = b$$

This is Required Complete Integral.

Type 1: PDE involving p and q only

PDE involving p and q only

The equation containing p and q only is of the form

$$f(\ p\ ,\ q\ ) = 0 \tag{8}$$

$$\frac{\partial f}{\partial x} = f_x = 0$$

$$\frac{\partial f}{\partial y} = f_y = 0$$

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Type 1: PDE involving p and q only Continue...

$$\frac{\partial f}{\partial z} = f_z = 0$$
$$\frac{\partial f}{\partial p} = f_p$$
$$\frac{\partial f}{\partial q} = f_q$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$
$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{0} = \frac{-dq}{0}$$

Special types of First Order Partial Differential Equations

Type 1: PDE involving p and q only Continue...

$$\text{Now, } EachRatio = \frac{-dp}{0}$$

$$\therefore \text{dp} = 0$$

$$\text{Integrating we get ,} \quad p = a$$

Using $p = a$ in equation (8), we obtain

$$q = \Phi(a)$$

using values of p and q in $p \ dx + q \ dy = dz$

$$a \ dx + \Phi(a) \ dy = dz$$

Integrating

$$ax + \Phi(a)y = z + b$$

$$ax + \Phi(a)y - z = b$$

My Inspiration Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Problem 8:

Solve the PDE $p + q = pq$

Solution

Let

$$f = p + q - pq = 0 \qquad (9)$$

It contains only p and q.

$\therefore$ p = a = constant

$\therefore$ $p = a = constant$

Using $p = a$ in equation (9)

$\therefore$ a + q = aq

Solution of Problem 8 Continue...

$$\therefore aq - q = a$$

$$\therefore q(a - 1) = a$$

$$\therefore q = \frac{a}{a-1}$$

$$q = \frac{a}{a-1}$$

Consider, $p\ dx + q\ dy = dz$

$$a\ dx + \frac{a}{a-1}\ dy = dz$$

Integrating we get, $ax + \dfrac{ay}{a-1} = z$

$a(a-1)x + ay = (a-1)z$... Required Solution

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Type 2: PDE not involving independent variables x and y

Type 2:PDE not involving independent variables x and y

The equation containing p and q only is of the form

$$f(\ p\ ,\ q\ ,\ z\ ) = 0 \tag{10}$$

$$\frac{\partial f}{\partial x} = f_x = 0$$
$$\frac{\partial f}{\partial y} = f_y = 0$$

Type 2: PDE not involving independent variables x and y

$$\frac{\partial f}{\partial z} = f_z$$
$$\frac{\partial f}{\partial p} = f_p$$
$$\frac{\partial f}{\partial q} = f_q$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$
$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{pf_z} = \frac{-dq}{qf_z}$$

Special types of First Order Partial Differential Equations

Type 2: PDE not involving independent variables x and y Continue

$$\implies \frac{-dp}{pf_z} = \frac{-dq}{qf_z}$$

$$\implies \frac{-dp}{p} = \frac{-dq}{q}$$

Integrating we get,

$$\ln p = \ln q + \ln a$$

$$\ln p = \ln aq$$

$$p = aq$$

Using $p = a$ in equation (10), we obtain expression for q using values of p and q in $p \ dx + q \ dy = dz$

Integrating we get required solution

Type 2: PDE not involving independent variables

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Problem 9:

Solve the PDE $p^2z^2 + q^2 = 1$

Solution

It is of the form

$$\text{f(p, q, z)} = p^2z^2 + q^2 - 1 = 0 \tag{11}$$

$$\text{Put} p = aq \text{ in (11) then } a^2q^2z^2 + q^2 = 1$$

$$\implies \text{q}^2(a^2z^2 + 1) = 1 \implies \text{q}^2 = \frac{1}{a^2z^2 + 1}$$

$$q = \frac{1}{\sqrt{a^2z^2 + 1}}$$

Solution of Problem 9 Continue...

Using value of q in $p = aq$

$$p = \frac{a}{\sqrt{a^2z^2+1}}$$

Consider, $p\ \ dx + q\ \ dy = dz$

$$\frac{a}{\sqrt{a^2z^2+1}}\ dx + \frac{1}{\sqrt{a^2z^2+1}}\ dy = dz$$

$$adx + dy = \sqrt{a^2z^2+1}dz$$

Integrating we get,

$$ax + y = a\int\sqrt{z^2+\frac{1}{a^2}}$$

$$ax + y = a\left[\frac{z}{z}\sqrt{z^2+\frac{1}{a^2}} + \frac{1}{2a^2}\ln\left(z + +\sqrt{z^2+\frac{1}{a^2}}\right)\right]$$

This is Required Solution

Type 2: PDE not involving independent variables

Problem 10:

Solve the PDE $pq + q^3 = 3pzq$

Solution

It is of the form

$$f(p, q, z) = pq + q^3 - 3pzq = 0 \tag{12}$$

$$\text{Put } p = aq \text{ in (12) then } aq^2 + q^3 = 3aq^2z$$
$$\implies q^2(a + q) = 3aq^2z$$
$$\implies (a + q) = 3az$$
$$q = 3az - a = a(3z - 1)$$

Type 2: PDE not involving independent variables

Solution of Problem 10 Continue...

Using value of q in $p = aq$

$$p = a^2(3z - 1)$$

Consider, $p \; dx + q \; dy = dz$

$$a^2(3z-1) \; dx + a(3z-1) \; dy = dz$$

$$a^2 dx + a dy = \frac{dz}{3z-1}$$

Integrating we get,

$$a^2 x + ay = \frac{\ln(3z-1)}{3} + b$$

$$a^2 x + ay = \frac{\ln(3z-1)}{3} + b$$

This is Required Solution

Type 2: PDE not involving independent variables

Problem 11:

Find the complete integral of $z^2(p^2z^2 + q^2) = 1$

Solution

It is of the form

$$f(p, q, z) = z^2(p^2z^2 + q^2) - 1 = 0 \tag{13}$$

Put $p = aq$ in (13) then $z^2(a^2q^2z^2 + q^2) = 1$

$$\implies z^2q^2(a^2z^2 + 1) = 1 \implies q^2 = \frac{1}{z^2(a^2z^2 + 1)}$$

$$q = \frac{1}{z\sqrt{a^2z^2 + 1}}$$

Solution of Problem 11 Continue...

Using value of q in $p = aq$

$$p = \frac{a}{z\sqrt{a^2z^2+1}}$$

Consider, $p\ dx + q\ dy = dz$

$$\frac{a}{z\sqrt{a^2z^2+1}}\ dx + \frac{1}{z\sqrt{a^2z^2+1}}\ dy = dz$$

$$adx + dy = z\sqrt{1+a^2z^2}dz$$

Integrating we get,

$$ax + y = \int z\sqrt{1+a^2z^2}dz + b$$

Solution of Problem 3 Continue...

For integration, we use substitution method

Put $1 + a^2z^2 = t$

$$\implies 2a^2 z dz = dt$$

$$\implies z\ dz = \frac{dt}{2a^2}$$

$$ax + y - b = \int \sqrt{t}\frac{dt}{2a^2}$$

$$ax + y - b = \frac{1}{2a^2}\frac{t^{\frac{3}{2}}}{\frac{3}{2}}$$

$$3a^2(ax + y - b) = (a^2z^2 + 1)^{\frac{3}{2}}$$

This is Required Solution.

Problem 12:

Find the complete integral of $q^2 = z^2p^2(1 - p^2)$

Solution

It is of the form

$$f(p, q, z) = q^2 - z^2p^2(1 - p^2) = 0 \qquad (14)$$

Put $p = aq$ in (14) then $q^2 = z^2a^2q^2(1 - a^2q^2)$

$$\implies 1 - a^2q^2 = \frac{1}{a^2z^2} \implies 1 - \frac{1}{a^2z^2} = a^2q^2$$

$$\implies \frac{a^2z^2 - 1}{a^2z^2} = a^2q^2$$

Solution of Problem 12 Continue...

$$\therefore \quad \mathrm{q}^2 = \frac{a^2z^2 - 1}{a^4z^2}$$

$$q = \frac{\sqrt{z^2a^2 - 1}}{a^2z}$$

Using value of q in $p = aq$

$$p = \frac{a\sqrt{z^2a^2 - 1}}{a^2z}$$

$$p = \frac{\sqrt{z^2a^2 - 1}}{az}$$

Consider, $p\ dx + q\ dy = dz$

$$\frac{\sqrt{z^2a^2 - 1}}{az}\ dx + \frac{\sqrt{z^2a^2 - 1}}{a^2z}\ dy = dz$$

$$adx + dy = \frac{a^2z}{\sqrt{z^2a^2 - 1}}dz$$

Type 2: PDE not involving independent variables

Solution of Problem 12 Continue...

Integrating we get,

$$ax + y = \int \frac{a^2 z}{\sqrt{z^2 a^2 - 1}} dz + b$$

For integration, we use substitution method

Put $a^2 z^2 - 1 = t$

$$\Longrightarrow 2a^2 zdz = dt \implies a^2 zdz = \frac{dt}{2}$$

$$ax + y = \int \sqrt{t} \frac{dt}{2\sqrt{t}} + b$$

$$ax + y = \frac{2\sqrt{t}}{2}$$

$$ax + y = \sqrt{a^2 z^2 - 1} + b$$

This is Required Solution.

Special types of First Order Partial Differential Equations

Type 3: Separable Form

Type 3: Separable Form

The partial differential equation is said to be separable if it can be written in the form

$$f(\ x\ ,\ p\ ) = g(\ y\ ,\ q\ ) \tag{15}$$

$$\text{Let } F = f(x,p) - g(y,q) = 0$$
$$\frac{\partial F}{\partial x} = F_x = f_x$$
$$\frac{\partial F}{\partial y} = F_y = -g_x$$

Special types of First Order Partial Differential Equations

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Type 3: Separable Form continue...

$$\frac{\partial F}{\partial z} = F_z = O$$

$$\frac{\partial F}{\partial p} = F_p = f_p$$

$$\frac{\partial F}{\partial q} = F_q = -g_q$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{p\,f_p - q\,g_q} = \frac{-dp}{f_x} = \frac{-dq}{-g_y}$$

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Type 3: Separable Form Continue ...

$$\text{Consider, } \frac{dx}{f_p} = \frac{-dp}{f_x}$$

$$\therefore \text{f}_x dx + f_p dp = 0$$

$$\therefore \text{d[f(x , p)]} = 0$$

Integrating we get,

$$f(x, p) = a$$

Similarly using this in (15)

$$g(y, q) = a$$

obtain expression for p and q from above two equations

using values of p and q in $p\ dx + q\ dy = dz$

Integrating we get required solution.

Type 3: Separable Form

Problem 13:

Find the complete integral of $p^2 + q^2 = x + y$

Solution

Given equation is

$$p^2 + q^2 = x + y$$

$$\therefore \mathrm{p}^2 - x = y - q^2$$

It is of the form

$$f(x, p) = g(y, q)$$

$$\text{Let } f(x, p) = p^2 - x = a$$

$$\therefore \mathrm{p}^2 = a + x$$

$$p = \sqrt{x + a}$$

Type 2: PDE not involving independent variables

Solution of Problem 13 Continue...

Similarly, $g(y, q) = y - q^2 = a$

$\therefore \ q^2 = y - a$

$q = \sqrt{y - a}$

Consider, $p \ dx + q \ dy = dz$

$\sqrt{x + a} \ dx + \sqrt{y - a} \ dy = dz$

$(x + a)^{\frac{1}{2}} dx + (y - a)^{\frac{1}{2}} dy = dz$

Integrating we get,

$$\frac{(x + a)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(y - a)^{\frac{3}{2}}}{\frac{3}{2}} = z + b$$

$$(x + a)^{\frac{3}{2}} + (y - a)^{\frac{3}{2}} = \frac{3}{2}(z + b)$$

This is required solution.

Type 3: Separable Form

Problem 14:

Find the complete integral of $p^2y(1+x^2) = qx^2$

Solution

Given equation is

$$p^2y(1+x^2) = qx^2$$

$$\therefore \mathrm{p}^2\left(\frac{1+x^2}{x^2}\right) = \frac{q}{y}$$

It is of the form

$$f(x,p) = g(y,q)$$

Let $f(x,p) = p^2\left(\frac{1+x^2}{x^2}\right) = a$

Type 3: Separable Form

Solution of Problem 14 Continue...

$$\therefore\ \mathrm{p}^2 = a\left(\frac{x^2}{1+x^2}\right)$$

$$p = x\sqrt{\frac{a}{1+x^2}}$$

$$\text{Similarly, } g(y,q) = \frac{q}{y} = a$$

$$q = ay$$

$$\text{Consider, } p\ dx + q\ dy = dz$$

$$\sqrt{\frac{a}{1+x^2}}\ xdx + ay\ dy = dz$$

$$\sqrt{a}\frac{xdx}{\sqrt{1+x^2}} + aydy = dz$$

Type 3: Separable Form

Solution of Problem 14 Continue...

Integrating we get,

$$\sqrt{a}\sqrt{1+x^2} + a\frac{y^2}{2} = z + b$$

$$2\sqrt{a}\sqrt{1+x^2} + ay^2 = 2z + b$$

This is required solution.

Type 4: Claraut's Equation:

Type 4: Claraut's Equation:

The partial differential equation is of the form $z = px + qy + f(p, q)$ Where x and y are independent and z is dependent variable called as Claraut's Equation.

$$\text{Now let } F = px + qy + f(x, p) - z = 0$$

$$\frac{\partial F}{\partial x} = F_x = p$$

$$\frac{\partial F}{\partial y} = F_y = q$$

Type 4: Claraut's Equation continue...

$$\frac{\partial F}{\partial z} = F_z = -1$$

$$\frac{\partial F}{\partial p} = F_p = x + f_p$$

$$\frac{\partial F}{\partial q} = F_q = y + f_q$$

Charpit's Auxiliary equation is,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p\,f_p + q\,f_q} = \frac{-dp}{f_x + p\,f_z} = \frac{-dq}{f_y + q\,f_z}$$

$$\frac{dx}{x + f_p} = \frac{dy}{y + f_q} = \frac{dz}{xp + pf_p + yq + qf_q} = \frac{-dp}{p - p} = \frac{-dq}{q - q}$$

Special types of First Order Partial Differential Equations

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Type 4: Claraut's Equation continue...

$$\text{Consider, Each Ratio} = \frac{dp}{0}$$

$$\therefore \text{dp} = 0$$

Integrating we get,

$$p = a$$

$$\text{Now Consider, Each Ratio} = \frac{dq}{0}$$

$$\therefore \text{dq} = 0$$

Integrating we get,

$$q = b$$

using values of p and q in $z = px + qy + f(p, q)$

$$z = ax + by + f(a, b)$$

Integrating we get required solution.

Type 4: Claraut's Equation:

Problem 15:

Find the complete integral of $(p+q)(z-px-qy)=1$

Solution

Given equation is

$$z - px - qy = \frac{1}{p+q}$$

$$z = px + qy + \frac{1}{p+q}$$

It is of the Claraut's Equation form

Hence its solution is,

$$z = ax + by + \frac{1}{a+b}$$

where a and b are constants.