Practical No 3 : Condition of Compatibility of First Order Partial Differential Equations and Some Problem

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Compatible Differential Equations

Let $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ be first order partial differentiable equations.If every solution of $f = 0$ is also solution of $g = 0$ and

Jacobian $J = \frac{\partial(f, g)}{\partial(p, q)} \neq 0$

then this two equations f and g are said to be Compatiable.

Compatible Differential Equations

Theorem :

Show that the condition for $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$ compatible is $[f, g] = 0$
i.e. $\frac{\partial(f, g)}{\partial(x, p)} + \frac{\partial(f, g)}{\partial(y, q)} + p\frac{\partial(f, g)}{\partial(z, p)} + q\frac{\partial(f, g)}{\partial(z, q)} = 0$

Proof

Let

$$f(x, y, z, p, q) = 0 \tag{1}$$

$$g(x, y, z, p, q) = 0 \tag{2}$$

Compatible Differential Equations

Proof of Theorem Continue...

be first order partial differential equations. From (1) and (2) we obtain

$$p = \Phi(x, y, z), \quad q = \Psi(x, y, z)$$

The condition that equations (1) and (2) should be compatible reduces to $p\ dx + q\ dy = dz$ is integrable.

$$\therefore \quad \Phi\ dx + \Psi\ dy - dz = 0 \tag{3}$$

is integrable.

Let $\bar{X} = (\Phi, \Psi, -1)$ then $\bar{X}.Curl\bar{X} = 0$

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Proof of Theorem Continue...

$$\text{Now, } Curl\bar{X} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \Phi & \Psi & -1 \end{vmatrix}$$

$$= (0 - \frac{\partial \Psi}{\partial z})\hat{i} - (0 - \frac{\partial \Phi}{\partial z})\hat{j} + (\frac{\partial \Psi}{\partial x} - \frac{\partial \Phi}{\partial y})\hat{k}$$

$$= -\frac{\partial \Psi}{\partial z}\hat{i} + \frac{\partial \Phi}{\partial z}\hat{j} + (\frac{\partial \Psi}{\partial x} - \frac{\partial \Phi}{\partial y})\hat{k}$$

$$\bar{X}.Curl\bar{X} = (\Phi\hat{i}, \Psi\hat{j}, -1\hat{k}).[-\frac{\partial \Psi}{\partial z}\hat{i} + \frac{\partial \Phi}{\partial z}\hat{j} + (\frac{\partial \Psi}{\partial x} - \frac{\partial \Phi}{\partial y})\hat{k}]$$

$$-\Phi\frac{\partial \Psi}{\partial z} + \Psi\frac{\partial \Phi}{\partial z} - \frac{\partial \Psi}{\partial x} + \frac{\partial \Phi}{\partial y} = 0$$

Proof of Theorem Continue...

$$\Psi \frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial y} = \Phi \frac{\partial \Psi}{\partial z} + \frac{\partial \Psi}{\partial x} \tag{4}$$

Differentiate (1) w.r.t x and z,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial x} = 0$$

$$\therefore \quad \mathrm{f}_x + f_p \Phi_x + f_q \Psi_x = 0 \tag{5}$$

$$\text{and } \frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}.\frac{\partial p}{\partial z} + \frac{\partial f}{\partial q}.\frac{\partial q}{\partial z} = 0$$

Compatible Differential Equations

Proof of Theorem Continue...

$$\therefore \quad f_z + f_p\Phi_z + f_q\Psi_z = 0 \qquad (6)$$

Multiply equation (6) by Φ then add it to equation (5)

$$(f_x + \Phi f_z) + f_p(\Phi_x + \Phi\Phi_z) + f_q(\Psi_x + \Phi\Psi_z) = 0 \qquad (7)$$

Differentiate (2) w.r.t x and z and as above, we get,

$$(g_x + \Phi g_z) + g_p(\Phi_x + \Phi\Phi_z) + g_q(\Psi_x + \Phi\Psi_z) = 0 \qquad (8)$$

Compatible Differential Equations

Proof of Theorem Continue...

Multiply equation (7) by g_p and (8) by f_p then take (7)-(8)

$$g_p(f_x + \Phi f_z) + g_p f_p(\Phi_x + \Phi\Phi_z) + g_p f_q(\Psi_x + \Phi\Psi_z) - f_p(g_x + \Phi g_z) - f_p g_p(\Phi_x + \Phi\Phi_z) - f_p g_q(\Psi_x + \Phi\Psi_z) = 0$$

$$g_p(f_x + \Phi f_z) + g_p f_q(\Psi_x + \Phi\Psi_z) - f_p(g_x + \Phi g_z) - f_p g_q(\Psi_x + \Phi\Psi_z) = 0$$

$$g_p(f_x + \Phi f_z) - f_p(g_x + \Phi g_z) + (\Psi_x + \Phi\Psi_z)(g_p f_q - f_p g_q) = 0$$

$$\Phi(g_p f_z - f_p g_z) + (f_x g_p - g_x f_p) + (\Psi_x + \Phi\Psi_z)(g_p f_q - f_p g_q) = 0$$

$$(f_x g_p - g_x f_p) + \Phi(g_p f_z - f_p g_z) = (\Psi_x + \Phi\Psi_z)(f_p g_q - (g_p f_q)$$

$$\therefore \quad \frac{\partial(f,g)}{\partial(x,p)} + \Phi\frac{\partial(f,g)}{\partial(z,p)} = J(\Psi_x + \Phi\Psi_z)$$

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Proof of Theorem Continue...

$$\therefore \quad (\Psi_x + \Phi\Psi_z) = \frac{1}{J} \quad \left[\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)}\right] \qquad (9)$$

Similarly diff. eq^{ns} (1) and (2) w.r.t y and z, we obtain

$$\therefore \quad (\Phi_y + \Psi\Phi_z) = \frac{-1}{J} \quad \left[\frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}\right] \qquad (10)$$

Using equation (4) we get,

$$\frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q} = 0$$

$[f,g] = 0$ It is condition for f and g are to be compatible.

Compatible Differential Equations

Problem 1:

Show that the PDE $xp = yq$ and $z(x\,p + y\,q) = 2xy$ are compatible. Find Solution

Solution:

Let

$$f = x\ p - y\ q = 0 \tag{11}$$

$$g = z(xp + yq) - 2xy = 0 \tag{12}$$

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Solution of Problem 1 Continue...

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}$$

$$= p(zx) - x(zp - 2y) = zpx - zpx + 2xy = 2xy$$

$$\frac{\partial(f,g)}{\partial(x,p)} = 2xy$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq - 2x & zy \end{vmatrix} = -2xy$$

$$\frac{\partial(f,g)}{\partial(y,q)} = -2xy$$

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Solution of Problem 1 Continue...

$$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial p}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p}\end{vmatrix}=\begin{vmatrix}0 & x\\ xp+yq & zx\end{vmatrix}=-x(xp+yq)$$

$$\frac{\partial(f,g)}{\partial(z,p)}=-x(xp+yq)$$

$$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial q}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q}\end{vmatrix}=\begin{vmatrix}0 & -y\\ xp+yq & zy\end{vmatrix}=y(xp+yq)$$

$$\frac{\partial(f,g)}{\partial(z,q)}=y(xp+yq)$$

Solution of Problem 1 Continue...

Condition for Compatible is

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q}$$

$$= 2xy - 2xy + p[-x(xp+yq)] + qy(xp+yq)$$

$$=\text{-x}^2p^2 - xypq + xypq + y^2q^2$$

$$=\text{y}^2q^2 - x^2p^2$$

$$[f,g] = 0$$

$\therefore$ f and g satisfies the condition of Compatibility.

$\therefore$ Given PDE are compatible.

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Solution of Problem 1 Continue...

By equation (11) $xp = yq$

Using this in (12) $z(xp + xp) = 2xy$

$2xpz = 2xy \implies p = \dfrac{y}{z}$

Using value of p in (11), we get

$x(\dfrac{y}{z}) = yq \implies q = \dfrac{x}{z}$

Using p and q in $p\ dx + q\ dy = dz$

$\therefore \dfrac{y}{z}dx + \dfrac{x}{z}dy = dz \implies ydx + x\ dy = zdz$

$\therefore d(xy) = zdz$

Integrating, we get

$xy = \dfrac{z^2}{2} + c$

$2xy - z^2 = c$ .... Required Solution.

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Problem 2:

Show that the PDE $x\,p - y\,q = x$ and $x^2\,p + q = xz$ are compatible. Hence find Solution

Solution:

Let

$$\text{f} = \text{x p - y q - x} = 0 \tag{13}$$

$$\text{g} = \text{x}^2\,p + q - xz = 0 \tag{14}$$

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Solution of Problem 2 Continue...

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p-1 & x \\ 2xp-z & x^2 \end{vmatrix} = x^2(p-1) - x(2xp-z)$$

$$\frac{\partial(f,g)}{\partial(x,p)} = x^2(p-1) - x(2xp-z)$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix} = -q$$

$$\frac{\partial(f,g)}{\partial(y,q)} = -q$$

Solution of Problem 2 Continue...

$$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial p}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p}\end{vmatrix}=\begin{vmatrix}0 & x\\ -x & x^2\end{vmatrix}=x^2$$

$$\frac{\partial(f,g)}{\partial(z,p)}=x^2$$

$$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial q}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q}\end{vmatrix}=\begin{vmatrix}0 & -y\\ -x & 1\end{vmatrix}=-xy$$

$$\frac{\partial(f,g)}{\partial(z,q)}=-xy$$

Compatible Differential Equations

Solution of Problem 2 Continue...

Condition for Compatible is

$$[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q}$$

$$= x^2(p-1) - x(2xz - z) + px^2 - q - qxy$$

$$=\text{-}x^2p - x^2 - 2x^2p + xz + x^2p - q - qxy$$

$$=(xz - q) - x^2 - qxy$$

$$= x^2p - x^2 - qxy \text{ .... by equation (14)}$$

$$= x(xp - yq) - x^2$$

$$= x.x - x^2$$

$$[f, g] = 0$$

$\therefore$ f and g satisfies the condition of Compatibility.

$\therefore$ Given PDE are compatible.

Solution of Problem 2 Continue...

Multiply equation (14) by y then add it in equation (13)
$(x + x^2y)\,p = x + xyz \implies x(1+xy)\quad p = x\,(1+yz)$

$$p = \frac{1+yz}{1+xy}$$

Using it in (13)

$$\frac{1+yz}{1+xy} - yq = x \implies \frac{1+yz}{1+xy} - x = yq$$

$$\implies yq = \frac{x + xyz - x - x^2y}{1+xy} \implies yq = \frac{y(xz - x^2)}{1+xy}$$

$$q = \frac{x(z-x)}{1+xy}$$

Solution of Problem 2 Continue...

Using p and q in $p\ dx + q\ dy = dz$

$$\therefore \quad \frac{1+yz}{1+xy}dx + \frac{x(z-x)}{1+xy}dy = dz$$

It is Pfaffian Differential Equation

Take x = constant $\implies \; dx = 0$

$\therefore$ (xz - x^2)$dy - (1+xy)dz = 0$

$\therefore$ x(z - x) dy - (1 + xy) dz = 0

Dividing throughout by $(z-x)(1+xy)$

$$\therefore \quad \frac{x}{1+xy}dy - \frac{dz}{z-x} = 0$$

Integrating, we get

$\ln(1+xy) - \ln(z-x) = \ln c_1$

Compatible Differential Equations

Solution of Problem 2 Continue...

$\therefore \quad \dfrac{1+xy}{z-x} = c_1$ Hence Solution is of the form

$\dfrac{1+xy}{z-x} = \Phi(x)$

Hence Required Solution is

$$\frac{1+xy}{z-x} = c$$