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Practical No 1.1 :Formation of Partial Differential equation

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11.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

These equations are formed either by the elimination of arbitrary constants or by the elimination of the arbitrary functions from a relation with one dependent variable and the rest two or more independent variables.

Observations: When p.d.e. formed by elimination of arbitrary constants

1. If the number of arbitrary constants are more than the number of independent variables in the given relations, the p.d.e. obtained by elimination will be of 2nd or higher order.
2. If the number of arbitrary constants equals the number of independent variables in the given relation, the p.d.e. obtained by elimination will be of order one.

Observations: When p.d.e. formed by elimination of arbitrary functions. When n is the number of arbitrary functions, we may get several p.d.e., but out of which generally one with two least order is selected.

e.g. $z = xf(y) + yg(x)$ involves two arbitrary functions, f and g. Here $\frac{\partial^4 z}{\partial x^2 \partial y^2} = 0$ ...(*i*)

and $xys = xp + yq - z$ (second order) ...(*ii*)

are the two p.d.e. are obtained by elimination of the arbitrary functions. However, 2nd equation being in lower in order to 1st is the desired p.d.e.

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Example 1: Form a partial differential equation by eliminating *a*, *b*, *c* from the relation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

[NIT Kurukshetra, 2003; KUK, 2000]

Solution: Clearly in the given equation a, b, c are three arbitrary constants and z is a dependent variable, depending on x and y.

We can write the given relations as:

$$f(x, y, z) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = 0 \qquad \ldots(1)$$

then differentiating (1) partially with respect to x and y respectively, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0, \quad \left(\text{Keeping } \frac{\partial y}{\partial x} = 0\right)$$

and

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0, \quad \left(\text{Keeping } \frac{\partial x}{\partial y} = 0\right)$$

or

$$\frac{2x}{a^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad c^2 x + a^2 z p = 0 \qquad \ldots(2)$$

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and $$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0 \quad\Rightarrow\quad c^2y+b^2zq=0 \qquad \ldots(3)$$

Again differentiating (2) with respect to x, we have

$$c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0$$

On substituting $\frac{c^2}{a^2}=-\frac{z}{x}\frac{\partial z}{\partial x}$ from (2) in above equation, we get

$$-\frac{z}{x}\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}=0$$

or $$xz\cdot\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\frac{\partial z}{\partial x}=0 \qquad \ldots(4)$$

Similarly, differentiating (3) partially with respect to y and substituting the value of $\frac{c^2}{b^2}$ from (3) in the resultant equation, we have

$$yz\frac{\partial^2 z}{\partial y^2}+y\left(\frac{\partial z}{\partial y}\right)^2-z\frac{\partial z}{\partial y}=0 \qquad \ldots(5)$$

Thus equations (4) and (5) are 'partial differential equations' of first degree and second order.

Example 2: Form partial differential equation from $z = x\, f_1(x + t) + f_2(x + t)$.

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Solution: Clearly z is a function of x and t

$$p = \frac{\partial z}{\partial x} = f_1(x+t) + x\, f_1'(x+t) + f_2'(x+t)$$

$$q = \frac{\partial z}{\partial t} = x\, f_1'(x+t) + f_2'(x+t)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f_1'(x+t) + x\, f_1''(x+t) + f_1'(x+t) + f_2''(x+t)$$

$$= 2\, f_1'(x+t) + x\, f_1''(x+t) + f_2''(x+t)$$

$$s = \frac{\partial}{\partial t}\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial x \partial t} = f_1'(x+t) + x\, f_1''(x+t) + f_2''(x+t)$$

$$t = \frac{\partial^2 z}{\partial t^2} = x\, f_1''(x+t) + f_2''(x+t)$$

Now $\quad (r + t) = 2\, f_1'(x+t) + 2x\, f_1''(x+t) + 2\, f_2''(x+t) = 2s$

or $\quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} - 2\frac{\partial^2 z}{\partial x \partial t} = 0$

Example 3: Form the partial differential equation by eliminating the arbitrary function,

$$F(x + y + z, x^2 + y^2 + z^2) = 0$$ **[KUK, 2004-05, 2003-04]**

Solution: Let $F\big((x+y+z),\ (x^2+y^2+z^2)\big)=0$ be $F(u,v)=0$ ...(1)

where $u=(x+y+z)$ and $v=(x^2+y^2+z^2)$ ...(2)

Clearly $F(u,\ v)=0$ is an implicit function.

$$\therefore \quad \left.\begin{aligned} 0 = F_x &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x} \quad \ldots(i) \\ 0 = F_y &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial y} \quad \ldots(ii) \end{aligned}\right\} \quad \ldots(3)$$

whereas $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}=(1+p) \quad \ldots(4)$$

$$\left(\text{since } \frac{\partial y}{\partial x}=0=\frac{\partial x}{\partial y} \text{ as } x \text{ and } y \text{ are two independent variables}\right)$$

and $$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}=(1+q) \quad \ldots(5)$$

Similarly, $\frac{\partial v}{\partial x} = (2x + 2zp)$ ...(6)

$$\frac{\partial v}{\partial y} = (2y + 2zq) \quad \ldots(7)$$

Thus, on substituting the values of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in equation (3), we get

$$\left.\begin{aligned} 0 &= \frac{\partial F}{\partial u}(1+p) + \frac{\partial F}{\partial v}(2x+2pz) \quad \ldots(i) \\ 0 &= \frac{\partial F}{\partial u}(1+q) + \frac{\partial F}{\partial u}(2y+2qz) \quad \ldots(ii) \end{aligned}\right\} \quad \ldots(8)$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$, we get

$$\begin{vmatrix} (1+p) & (2x+2pz) \\ (1+q) & (2y+2qz) \end{vmatrix} = 0 \quad \Rightarrow \quad p(y-z) + q(z-x) = (x-y)$$

which is the desired p.d.e.

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Example 4: Form the partial differential equation (by eliminating the arbitrary function) from: $F(xy + z^2, x + y + z) = 0$. **[NIT Kurukshetra, 2007; KUK, 2002-03]**

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Solution: Let $F(xy + z^2, x + y + z) = 0$ be $F(u, v) = 0$ ...(1)

where $u = xy + z^2$

and $v = x + y + z$ ...(2)

Clearly $F(u, v) = 0$ is an implicit relation, so that

$$\left.\begin{aligned} 0 = F_x &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} \quad \ldots(i) \\ 0 = F_y &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} \quad \ldots(ii) \end{aligned}\right\} \qquad \ldots(3)$$

whereas $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) = (y + 2zp) \qquad \ldots(4)$$

and $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) = (x + 2zp) \qquad \ldots(5)$$

Similarly, $$\frac{\partial v}{\partial x} = (1 + p) \qquad \ldots(6)$$

$$\frac{\partial v}{\partial y} = (1 + q) \qquad \ldots(7)$$

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On substituting the values of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ in equation (3), we get

$$\left.\begin{aligned} 0 &= \frac{\partial F}{\partial u}(y + 2pz) + \frac{\partial F}{\partial v}(1 + p) \\ 0 &= \frac{\partial F}{\partial u}(x + 2qz) + \frac{\partial F}{\partial v}(1 + q) \end{aligned}\right\} \qquad \ldots(8)$$

On eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$, we get

$$\begin{vmatrix} y + 2pz & 1 + p \\ x + 2qz & 1 + q \end{vmatrix} = 0$$

$\Rightarrow$ $p(2z - x) - (2z - y)q = (x - y)$ the desired partial differentiation equation.

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Example 5: Form partial differential equation from the relation

(*i*) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ (*ii*) $z = f_1(x + iy) + f_2(x - iy)$.

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Solution: (*i*) $$z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \quad \ldots(1)$$

$\therefore$ $$\frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right)\cdot\left(-\frac{1}{x^2}\right) \quad \ldots(2)$$

and $$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\cdot\left(\frac{1}{y}\right) \quad \ldots(3)$$

On eliminating of $2f'\left(\frac{1}{x} + \log y\right)$, we get (3) as

$$\frac{\partial z}{\partial y} = 2y + \left[-x^2\frac{\partial z}{\partial x}\right]\frac{1}{y}$$

$\Rightarrow$ $yq = 2y^2 - x^2 p;$ when $\frac{\partial z}{\partial x} = p$ and $\frac{\partial z}{\partial y} = q.$

(*ii*) Given $z = f_1(x + iy) + f_2(x - iy)$ ...(1)

$$\frac{\partial z}{\partial x} = f_1'(x + iy) + f_2'(x - iy) \quad \ldots(2)$$

$$\frac{\partial z}{\partial y} = i\, f_1'(x + iy) - i\, f_2'(x - iy) \quad \ldots(3)$$

Similarly $\frac{\partial^2 z}{\partial x^2} = f_1''(x + iy) + f_2''(x - iy)$ ...(4)

$$\frac{\partial^2 z}{\partial y^2} = i^2 f_1''(x + iy) + i^2 f_2''(x - iy) \quad \ldots(5)$$

$\therefore$ $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0;$ where $i^2 = -1$

Example 6: Form partial differential equations from the solutions

(*i*) $z = f(x) + e^{y} g(x)$

(*ii*) $z = \frac{1}{r}[F(r - at) + F(r + at)]$ **[NIT Kurukshetra, 2008]**

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Solution: (*i*): Given $z = f(x) + e^y g(x)$

$$\therefore \quad \frac{\partial z}{\partial y} = e^y g(x), \text{ Keeping } g(x) \text{ as constant.}$$

and $$\frac{\partial^2 z}{\partial y^2} = e^y g(x), \text{ (On differentiating again with respect to } y)$$

Thus $$\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2}$$

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(*ii*) Given $$z = \frac{1}{r}[F(r - at) + F(r + at)] \qquad \ldots(1)$$

$$\frac{\partial z}{\partial t} = \frac{1}{r}\left[F'(r - at) \cdot -a + F'(r + at) \cdot a\right] \qquad \ldots(2)$$

$$\frac{\partial^2 z}{\partial t^2} = \frac{a^2}{r}\left[F''(r - at) + F''(r + at)\right] \qquad \ldots(3)$$

$$\frac{\partial z}{\partial r} = \frac{1}{r}\left[F'(r - at) + F'(r + at)\right] - \frac{1}{r^2}\left[F(r - at) + F(r + at)\right] \qquad \ldots(4)$$

$$\Rightarrow \quad \frac{\partial z}{\partial r} = \frac{1}{r}\left[F'(r - at) + F'(r + at)\right] - \frac{z}{r}$$

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$$\frac{\partial^2 z}{\partial r^2} = \frac{1}{r}[F''(r-at) + F''(r+at)] - \frac{1}{r^2}[F'(r-at) + F'(r+at)]$$
$$- \frac{1}{r^2}[F'(r-at) + F'(r+at)] + \frac{2}{r^3}[F(r-at) + F(r+at)]$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial r^2} = \frac{1}{r}[F''(r-at) + F''(r+at)] - \frac{2}{r^2}[F'(r-at) + F'(r+at)] + \frac{2}{r^3}[F(r-at) + F(r+at)] \quad \ldots(5)$$

On using (1), (3), (4) in (5), we get

$$\frac{\partial^2 z}{\partial r^2} = \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2} - \frac{2}{r}\left[\frac{\partial z}{\partial r} + \frac{z}{r}\right] + \frac{2}{r^2} z$$

$$\frac{\partial^2 z}{\partial r^2} + \frac{2}{r}\frac{\partial z}{\partial r} = \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2} \quad \text{or} \quad \frac{a^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial z}{\partial r}\right) = \frac{\partial^2 z}{\partial t^2}$$ is the desired p.d.e.