4. Nov. 2022

Time: 2.5 Hours Total Marks: 75

N. B. 1) All questions are compulsory.
2) Use of a simple calculator is allowed.
3) Figures to the right indicate marks.

Q. 1 A) Attempt **any one** from the following. **(8)**

i) If U is an open set in $\mathbb{R}^2$ containing the rectangle $[a,b] \times [c,d]$ and $f: U \to \mathbb{R}$ is continuously differentiable function then show that $g'(x) = \int_c^d \frac{\partial f}{\partial x}(x,y)dy$ where $g(x) = \int_c^d f(x,y)dy,\ \forall x \in [a,b]$.

ii) State and prove the Fubini's Theorem for a rectangular domain in $\mathbb{R}^2$.

B Attempt **any two** from the following. **(12)**

i) Prove that every continuous function defined on a rectangular domain R in $\mathbb{R}^2$ is integrable.

ii) Using cylindrical co-ordinates find the volume of the solid region S in $\mathbb{R}^3$ which is bounded by the paraboloid $x^2 + y^2 = 10 - z$ and the plane $z = 1$.

iii) Evaluate $\iiint_S (x^2 + y^2 + z^2)^{\frac{1}{2}} dxdydz$ where S is region in $\mathbb{R}^3$ bounded by two co-centric spheres with centre at the orgin and radii 1 and 3.

Q.2 A) Attempt **any one** from the following. **(8)**

i) Define a parameterized curve in $\mathbb{R}^n$. When do you say that two parameterized curves in $\mathbb{R}^n$ are equivalent? Show that two equivalent parameterized curves have essentially the same image set. Show that the converse of this is not true by considering the curves $\alpha(t) = (\cos t, \sin t)\,;\ 0 \le t \le 2\pi$ and $\beta(t) = (\sin t, \cos t);\ 0 \le t \le 2\pi$.

ii) State and prove the Green's theorem for a rectangle.

B Attempt **any two** from the following. **(12)**

i) Evaluate $\int_C 2xydx + x^2zdy + x^2ydz$, where C is a straight line joining $(1,1,1)$ to $(1,2,4)$.

ii) Calculate the work done in the moving the particle from the point $P = (2,-1)$ to the point $Q = (-4,2)$ for the force field $F(x,y) = (x^2 + 4xy + 4y^2, 2x^2 + 8xy + 8y^2)$, showing first that it is conservative.

iii) Use Green's theorem to evaluate $\oint 4x^2\, y\, dx + 2y\, dy$ over closed curve C, where C is the boundary of the triangle with vertices $(0,0), (1,2)$ and $(0,2)$.

Q.3. A) Attempt **any one** from the following. **(8)**

i) State and prove the Gauss Divergence theorem for a cube.

ii) State and Prove the Stoke's theorem.

B Attempt **any two** from the following. **(12)**

i) Using Stoke's Theorem evaluate the line integral $\oint_C xydx + x^2ydy$ taken around the square C with vertices $(1,0), (-1,0), (0,1)$ and $(0,-1)$.

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ii) Evaluate the surface integrals of vector field $F(x,y,z) = (x,y,z)$ and S is the piece of the cylinder with parameterization $r(x,y) = (\cos x, \sin x, y)$ where $(x,y) \in \left[0, \frac{\pi}{2}\right] \times [0,1]$.

iii) Let $S = \bar{r}(T)$ be a smooth parametric surface in uv plane. Define area of S. If S is represented by an equation $z = f(x,y)$ then show that area of S is given by

$$\iint_T \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\ dxdy$$

where T is projection of S on xy –plane.

Q. 4 A Attempt **any three** questions from the following. **(15)**

i) Evaluate $\iint_S dxdy$ where S is the region bounded by the curves $xy = 4, xy = 8, xy^3 = 5, xy^3 = 15$.

ii) Evaluate $\iiint_S e^{(x^2+y^2+z^2)^{3/2}}\ dxdydz$ where S is the unit sphere centered at origin by using spherical coordinates.

iii) Evaluate line integral of scalar field $f(x,y,z) = x + y + z$ along the line segment from (1, 2, 3) to (0, -1, 1).

iv) Calculate the work done in the moving the particle from the point P to the point Q by the force field $F(x,y) = \left(y(e^{xy}+1), x(e^{xy}+1)\right)$; showing first that it is conservative where $P(1,0)$, $Q(1,1)$.

v) If S and C satisfy hypothesis of Stoke's Theorem and f, g have continuous second order partial derivatives. Prove with usual notations
i) $\int_C (f\nabla g).dr = \iint_S (\nabla f \times \nabla g).\bar{n}dS$ ii) $\int_C (f\nabla g) + g\nabla f).\ dr = 0$.

vi) Prove the following identities, assuming S and V satisfy the conditions of the Divergence Theorem and components of F have continuous partial derivatives, $\bar{n}$ is unit outward normal.
i) $|V| = \frac{1}{3}\iint_S \bar{r}.\ \bar{n}dS$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $|V|$ = volume of V.
ii) $\iint_S curlF.\bar{n}dS = 0$.

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Paper / Subject Code: 24286 / Mathematics : Group Theory (R 2022)

5- Nov- 2022 | 10.30 AM to 1.00 PM.

Time: $2\frac{1}{2}$ hours **Total Marks:75**

Instructions:1) All questions are compulsory.

2) Figures to right indicate marks for respective sub questions.

Q.1 A Answer any ONE

i) Let G be a group and H be a non-empty subset of G. Prove that H is a subgroup of G if and only if $ab^{-1} \in H$ for all $a, b \in H$ 08

ii) Let G be a group, $a, b \in G$ such that $o(a) = m, o(b) = n$ and $ab = ba$. If $gcd(m, n) = 1$ then prove that $o(ab) = mn$. 08

B Answer any TWO

i) Let H and K be subgroups of a group G then prove that $H \cup K$ is also a subgroup of G if and only if either $H \subseteq K$ or $K \subseteq H$. 06

ii) Let G be a group. Prove that $\phi: G \to G$ defined as $\phi(x) = x^{-1}$ is homomorphism if and only if G is an abelian 06

iii) Let G be a group and $a \in G$ with $o(a) = n$ then, prove that $a^m = e$ if and only if $n|m$. 06

Q.2 A Answer any ONE

i) If H_1, H_2 are normal subgroups of groups G_1, G_2 respectively, then prove that $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$. Further prove that $\frac{G_1 \times G_2}{H_1 \times H_2}$ is isomorphic to $\frac{G_1}{H_1} \times \frac{G_2}{H_2}$. 08

ii) State and prove first isomorphism theorem of group (Fundamental Theorem of Group Homomorphism). 08

B Answer any TWO

i) Let G, G' be groups and $f: G \to G'$ be an onto homomorphism. Prove that: The kernel of f is a normal subgroup of G and image of f is a subgroup of G'. 06

ii) Prove that $f: S_n \to \{\pm 1\}$ defined as $f(\sigma) = \begin{cases} 1, & if\ \sigma\ is\ even \\ -1, & if\ \sigma\ is\ odd \end{cases}$ is a homomorphism where $\{\pm 1\}$ is a group under multiplication. Further show that A_n is normal in S_n. 06

iii) Let G be a group and H be a unique subgroup of G of given order. Then prove that H is a normal subgroup of G. 06

Q.3 A Answer any ONE

i) Define cyclic subgroup and normal subgroup. If a cyclic subgroup H of a group G is normal in G, show that every subgroup of H is normal in G. 08

ii) Let G be a finite cyclic group of order n. Show that for every positive divisor d of n, there exists a unique subgroup of order d. 08

B Answer any TWO

i) Let G be a cyclic group of order 18 generated by a. Find all the generators of G. Further, find all the elements of order 9 in G.Clearly state the results used. 06

ii) Show that an infinite cyclic group generated by a has exactly two generators a and a^{-1} 06

iii) Show that a group of prime order is cyclic. Is converse true? Justify your answer. 06

Q.4 Answer any THREE

i) Let G be the group of functions from $\mathbb{R}$ to $\mathbb{R}^*$ under point wise multiplication. Let $H = \{ f \in G \mid f(1) = 1\}$. Prove that H is a subgroup of G. 05

ii) Let G be a group and $a \in G$. If $o(a) = mn$ then show that $o(a^m) = n$. 05

iii) List all left cosets of the subgroup $H = \{\overline{1}, \overline{11}\}$ of $U(30)$, the group of residue classes under multiplication modulo 30. 05

iv) Show that if H, N are subgroups of a group G and N is a normal subgroup of G, then prove that $H \cap N$ is normal in H. Give an example to show that $H \cap N$ need not be normal in G. 05

v) Find the number of elements in the cyclic subgroup of the group $\mathbb{C}^*$ (of non-zero complex numbers) generated by $1 + i$ 05

vi) Let μ_5 denotes the multiplicative group of the fifth roots of unity in $\mathbb{C}$. Determine the order of the cyclic subgroup of μ_5 generated by $e^{\frac{4i\pi}{5}}$ 05

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07 NOV 2022 | 10.30 am to 1.00 pm

Duration 2 ½Hrs

REVISED COURSE

Marks: 75

N.B. : (1) All questions are compulsory.
(2) Figures to the right indicate marks.

1. (a) Attempt ANY ONE from the following: (8)
   (i) Show that for a subset F of a metric space (X, d), the following statements are equivalent:
   (I) F is closed
   (II) F contains all its limit points.
   (ii) Show that in a metric space (X, d)
   (I) an arbitrary union of open sets is an open set.
   (II) a finite intersection of open sets is an open set.

   (b) Attempt ANY TWO from the following: (12)
   (i) Define an open ball $B(x, r)$ in a metric space (X, d) and show that every open ball is an open set.
   (ii) Let (X, d) be a metric space and $d_1 : X \times X \to \mathbb{R}$ be a metric defined as $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}, \forall x, y \in X$. Show that d and d_1 are equivalent metrics on X.
   (iii) Show that in a discrete metric space (X, d), every subset is both open and closed.

2. (a) Attempt ANY ONE from the following: (8)
   (i) State and prove Density Theorem.
   (ii) Let (X, d) be a metric space and A be a subset of X. Show that $p \in X$ is a limit point of A if and only if there is a sequence of distinct points in A converging to p.

   (b) Attempt ANY TWO from the following: (12)
   (i) Prove that a subspace (Y, d) of complete metric space (X, d) is complete if and only if Y is closed.
   (ii) If $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function such that f takes only rational values then show that f is a constant function.
   (iii) Show that $S = \{x \in \mathbb{Q} : 3 < x^2 < 5\}$ is both open and closed in the subspace $\mathbb{Q}$ of $\mathbb{R}$ with usual metric.

3. (a) Attempt ANY ONE from the following: (8)
   (i) Consider the metric space $(\mathbb{R}, d)$ where d is usual metric, Prove that if $a, b \in \mathbb{R}$ where $a < b$, then Prove that $[a, b]$ satisfies Heine-Borel property.
   (ii) Define compact subset of a metric space. Show that a compact subset of a metric space is closed. Give an example to show that every closed subset need not be compact.

   (b) Attempt ANY TWO from the following: (12)
   (i) Consider the metric space $(C[a, b], \| \|_\infty)$, where $\|f\|_\infty = \sup \{|f(t)| : t \in [a, b]\}$. Show that the open cover $\{B(0, n)\}_{n \in \mathbb{N}}$ of $C[a, b]$ has no finite subcover. (0 being the constant zero function).

(ii) Determine if $D = \{(x,y) \in \mathbb{R}^2 : |y| \leq 2\}$ in $(\mathbb{R}^2, d)$ where d is Euclidean distance, is compact in $\mathbb{R}^2$.

(iii) If A, B are compact subsets of any metric space, prove that $A \cup B$ is also compact.

4. Attempt ANY THREE from the following: (15)

(a) Define an open set in a metric space (X, d). Let $A \subseteq X$. Show that A is open if and only if $A = A^{\circ}$ (Interior of A).

(b) Let d_1, d_2 be metrics on a non-empty set X. Define $d : X \times X \longrightarrow \mathbb{R}$ as $d(x,y) = \max\{d_1(x,y), d_2(x,y)\}$. Show that d is a metric on X.

(c) Prove that the metric space $(\mathbb{R}^2, d_1)$ is complete where the metric d_1 is given by $d_1((x_1,y_1),(x_2,y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$.

(d) Show that the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = (x-a)^2(x-b)^2 + x$, takes the value $\frac{a+b}{2}$ for some value of $x \in \mathbb{R}$. (distance in $\mathbb{R}$ being usual)

(e) Is it true that interior and closure of a compact set is compact? Justify.

(f) Show that $[0,1) \subset (\mathbb{R}, d)$ with d usual distance on $\mathbb{R}$, is not sequentially compact.

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3. (a) Attempt any One of the following. (8)

(i) Consider the quasi-linear equation $P(x,y,z)\,p+Q(x,y,z)\,q=R(x,y,z)$ where P,Q and R are continuously differentiable functions on a domain $\Omega \subseteq \mathbb{R}^3$. If $S: z=u(x,y)$ is the surface obtained by taking the union of characteristic curves of the given p.d.e. where $u(x,y)$ is a continuously differentiable function then prove that S is the integral surface of the p.d.e.

(ii) Write a short note on the characteristic strip and characteristic curve for a non linear first order partial differential equation $f(x,y,z,p,q)=0$.

(b) Attempt any Two of the following. (12)

(i) Find the solution of the initial value problem for the quasi-linear equation $p-z\,q=-z$ for all y and $x>0$ with the initial data curve $C: x_0(s)=0, y_0(s)=s, z_0(s)=-2s, \quad -\infty<s<\infty$.

(ii) Find the initial strip for $z=\frac{1}{2}(p^2+q^2)+(p-x)(q-y)$ passing through the $x-$axis.

(iii) Find the characteristics differential equations and the characteristic strips of $pq=xy$.

4. Attempt any Three of the following. (15)

(a) Find a partial differential equation satisfied by $z=f(x^2+y^2)$.

(b) Show that $(x-a)^2+(y-b)^2+z^2=1$ is a solution of $z^2(1+p^2+q^2)=1$.

(c) Show that the partial differential equations $xp=yq$ and $z(xp+yq)=2xy$ are compatible.

(d) Find a complete integral of the partial differential equation $z=px+qy+pq$.

(e) For the partial differential equation $x^3\,p+y(3x^2+y)\,q=z(2x^2+y)$ and the initial data curve $x_0(s)=1, y_0(s)=s, z_0(s)=s^2+s$, find the value of

$$\frac{dy_0}{ds}P\Big(x_0(s),y_0(s),z_0(s)\Big)-\frac{dx_0}{ds}Q\Big(x_0(s),y_0(s),z_0(s)\Big)$$

where $x^3\,p+y(3x^2+y)\,q=z(2x^2+y)$ is compared with $P(x,y,z)\,p+Q(x,y,z)\,q=R(x,y,z)$.

(f) Show that the characteristic curves of $z\,p+q=0$ containing the initial data curve

$$C: x_0(s)=s, y=y_0(s)=0, z_0(s)=f(s) \text{ where } f(s)=\begin{cases}1 & \text{if } s\le 0,\\ 1-s & \text{if } 0\le s\le 1,\\ 0 & \text{if } s\ge 1.\end{cases}$$

are straight lines given by $x=y\,f(s)+s$.

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TYBSC (09/11/2022) 10·30 AM – 01·00 PM

Duration: 2 ½ Hrs Marks: 75

N.B. : (1) All questions are compulsory.
(2) Figures to the right indicate marks.

1. (a) Attempt any One of the following: (8)

(i) Prove that the elimination of arbitrary function ϕ from the equation $\phi(u, v) = 0$, where u and v are functions of x, y and z (z is assumed to be a function of x and y), gives the partial differential equation

$$\frac{\partial(u,v)}{\partial(y,z)} * p + \frac{\partial(u,v)}{\partial(x,z)} * q = \frac{\partial(u,v)}{\partial(x,y)}.$$

(ii) If $z = F(x, y; a)$ is a one-parameter family of solutions of the partial differential equation $f(x, y, z, p, q) = 0$ where $p = z_x = F_x, q = z_y = F_y$, prove that the envelope of this family, if it exists, is also a solution of $f(x, y, z, p, q) = 0$.

(b) Attempt any Two of the following. (12)

(i) Find a partial differential equation satisfied by $z = f\left(\frac{y}{x}\right)$ where f is real valued function on $\mathbb{R} \times \mathbb{R}$.

(ii) Find the singular integral of $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0$ using the three equations : $f(x, y, z, p, q) = 0, f_p(x, y, z, p, q) = 0, f_q(x, y, z, p, q) = 0$.

(iii) Solve the Lagrange's partial differential equation $\tan x \; p + \tan y \; q = \tan z$.

2. (a) Attempt any One of the following: (8)

(i) (I) If $p = \phi(x, y, z)$ and $q = \psi(x, y, z)$ are obtained by solving $f(x, y, z, p, q) = 0, g(x, y, z, p, q) = 0$ for p and q then state the necessary and sufficient condition for the equation $dz = \phi(x, y, z)\, dx + \psi(x, y, z)\, dy$ to be integrable.

(II) Show that the first order partial differential equations $p = M(x, y)$ and $q = N(x, y)$ are compatible if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

(ii) State Charpit's Auxiliary equations and explain the Charpit's method to find a complete integral of a given partial differential equation $f(x, y, z, p, q) = 0$.

(b) Attempt any Two of the following. (12)

(i) Solve the compatible partial differential equations $xp - yq = x$ and $x^2p + q = xz$ and find a common solution.

(ii) Show that $dz = \phi(x, y, z)\, dx + \psi(x, y, z)\, dy$ is integrable if and only if

$$-\phi\, \psi_z + \psi\, \phi_z - \psi_x + \phi_y = 0.$$

(iii) Find a complete integral of $x^2p^2 + y^2q^2 - 4 = 0$.