Unit I : First Order Partial Differential Equation

Subject Teacher
Santosh Dhamone

Assistant Professor in Mathematics
Art's Commerce and Science College,Onde
Tal:- Vikramgad, Dist:- Palghar

ssdhamone@acscollegeonde.ac.in

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Contents

- Parametrically Defined Curve in $\mathbb{R}^2$:
  Parametrically Defined Curve in $\mathbb{R}^2$ is the plane curve Ç given by a function
  $f : D \longrightarrow \mathbb{R}^2, f(t) = (x(t), y(t))$ where $D \subseteq \mathbb{R}$.
  Usually, we express this by simply saying that C is the (parametrically defined) curve given by $(x(t), y(t)), t \in D$. For example, the rectangular hyperbola is the curve $(t, \frac{1}{t}), t \in \mathbb{R} \setminus \{0\}$.
- Parametrically Defined Curve in $\mathbb{R}^3$:
  A parametrically defined curve C in $\mathbb{R}^3$ is given by
  $\Big(x(t), y(t), z(t)\Big), t \in D$ where $D \subseteq \mathbb{R}$.
- Parametrically Defined Surface S in $\mathbb{R}^3$ is given by a function
  $f : D \longrightarrow \mathbb{R}^3, f(u, v) = \Big(x(u, v), y(u, v), z(u, v)\Big)$ where x, y, z are real valued functions on D, that is, $x, y, z :\longrightarrow \mathbb{R}$.

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We will need the following Chain Rules.

- If $z = g(v)$ and $v = f(x, y)$ and then z is a function of (x, y), and

$$\frac{\partial z}{\partial x} = \frac{dz}{dv}\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{dz}{dv}\frac{\partial v}{\partial y}$$

- If $z = f(x, y)$ and $x = x(t), y = y(t)$, then z is a function of t, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

- If $z = f(u, v)$ and if $u = u(x, y)$, $v = v(x, y)$, then z is a function of (x, y), and

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

PDE: Unit I : Useful Notations

- If z is a function of (x, y) then $\boxed{\dfrac{\partial z}{\partial x} = p, \quad \dfrac{\partial z}{\partial y} = q.}$
- If $u = u(x, y)$ and $v = v(x, y)$ then

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \text{Jacobian matrix of } u \text{ and } v \text{ w.r.t. } x \text{ and } y.$$

- $\det \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \text{Jacobian of } u \text{ and } v \text{ w.r.t. } x \text{ and } y.$

- Notation $\dfrac{\partial(u, v)}{\partial(x, y)} = \det \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$

PDE: Unit I : Definition of PDE, Order and Degree

- An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a partial differential equation.
- General format $f(x, y, z, p, q) = 0$
- For example $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy$

  That is,

$$p + q = z + xy$$

- Order of a partial differential equation is defined as the order of the highest order partial derivative occurring in the partial differential equation.
- Degree of a partial differential equation is defined as the power of the highest order partial derivative occurring in the partial differential equation after the equation has been made free from radicals and fractions so far as derivatives are concerned.

A first order partial differential equation $f(x, y, z, p, q) = 0$ is known as

Linear eqn	Semi-linear	Quasi-linear
linear in p, q and z	linear in p, q	linear in p, q
$P(x,y)p + Q(x,y)q$ $= R(x,y)z + S(x,y)$	$P(x,y)p + Q(x,y)q$ $= R(x,y,z)$	$P(x,y,z)p + Q(x,y,z)q$ $= R(x,y,z)$
For example $yx^2\ p + xy^2\ q$ $= x + y + x^2y^2z$	For example $yx^2\ p + xy^2\ q$ $= x + y + x^2y^2z^2$	For example $yx^2z\ p + xy^2z\ q$ $= x + y$

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Theorem 0.1

TYPE 1 The elimination of arbitrary function ϕ from the equation $\phi(u, v) = 0$, where u and v are functions of x, y and z (z is assumed to be a function of x and y), gives the partial differential equation

$$\frac{\partial(u,v)}{\partial(y,z)} * p + \frac{\partial(u,v)}{\partial(z,x)} * q = \frac{\partial(u,v)}{\partial(x,y)}.$$

Obtain a pde by eliminating the arbitrary function ϕ from $\phi(x+y+z, x^2+y^2-z^2) = 0$

Let $u(x,y,z) = x+y+z, v(x,y,z) = x^2+y^2-z^2$.

$\frac{\partial(u,v)}{\partial(y,z)}$	$\frac{\partial(u,v)}{\partial(z,x)}$	$\frac{\partial(u,v)}{\partial(x,y)}$
$\det\begin{pmatrix}1 & 1\\ 2y & -2z\end{pmatrix}$	$\det\begin{pmatrix}1 & 1\\ -2z & 2x\end{pmatrix}$	$\det\begin{pmatrix}1 & 1\\ 2x & 2y\end{pmatrix}$
$-2(y+z)$	$2(x+z)$	$2(y-x)$

The required pde is $\boldsymbol{(y+z)\,p - (x+z)\,q = x - y}$

Theorem 0.2

TYPE 2: Let $v = v(x, y)$ and $z = f(v)$ then $\dfrac{\partial(z, v)}{\partial(x, y)} = 0$ is a first order pde for z. That is, the pde is $\begin{vmatrix} p & q \\ v_x & v_y \end{vmatrix} = 0$

Example: Eliminate arbitrary function f from $z = f(x^2 - y^2)$ and obtain the corresponding pde.

Let $v = x^2 - y^2$. So $v_x = 2x, v_y = -2y$.

The partial differential is given by $\dfrac{\partial(z, v)}{\partial(x, y)} = 0$

That is, $v_y\ p - v_x\ q = 0$

Substituting we get, $(-2y)\ p - (2x)\ q = 0$. The p.d.e. is

$$\boldsymbol{y\ p + x\ q = 0}.$$

Unit I : Obtaining a PDE

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Using theorem 0.2 we can prove following results.

Equation	pde
$z = f(x^2 - y^2)$	$y\ p + x\ p = 0$
$z = f(x^2 + y^2)$	$y\ p - x\ q = 0$
$lx + my + nz = f(x^2 + y^2 + z^2)$	$\dfrac{l + np}{m + nq} = \dfrac{x + zp}{y + zq}$
$z = e^{ax+by} f(ax - by)$	$b\ p + a\ q = 2ab\ z$
$z = x^n f\left(\dfrac{y}{x}\right)$	$x\ p + y\ q = nz$

TYPE 3: Obtain a pde by the elimination of arbitrary constants:
Consider the equation

$$F(x, y, z, a, b) = 0$$

where a and b denote arbitrary constats.

- **Case 1**: No. of arbitrary constants $<$ No. of independent variables.
  For example: $z = ax + y$. So, $p = a, q = 1$.
  Substituting $a = p$ in the equation, we get one pde $\boldsymbol{z = x\,p + y}$.
  Note that $\boldsymbol{q = 1}$ is also a pde of $z = ax + y$.
- **Case 2**: No. of arbitrary constants $=$ No. of independent variables.
  then the elimination gives rise to a unique partial differential equation of order one.
- For example, $az + b = a^2x + y$.
  $\Longrightarrow a\ p = a^2$ and $a\ q = 1 \Longrightarrow a^2\ p *\ q = a^2$
- The pde is $\boldsymbol{p\ q = 1}$. Note: pde is not a linear diff. eqn.
- **Case 3**: No. of arbitrary constants $>$ No. of independent variables.
  For example $z = ax + by + cxy$. The elimination of arbitrary constants leads to a pde of order usually greater than one.

- Consider a first order partial differential equation in two unknowns x and y given be

$$f(x, y, z, p, q) = 0 \tag{1}$$

The solution $z = F(x, y; a, b)$ of (1) represents a surface in (x, y, z) space.
This surface is called an integral surface of the partial differential equation (1).

- A two parameter family of solutions $z = F(x, y; a, b)$ of the equation

$$f(x, y, z, p, q) = 0.$$

is called a **complete integral** of the equation $f(x, y, z, p, q) = 0$ if the rank of the matrix

$$M = \begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix} \text{ is two.}$$

Example 0.3

Consider $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0$.
Show that the two parameter family of $z = F(x, y; a, b)$ given by $z = ax + by + a^2 + b^2$ is a complete integral.

Solution:

$$z = ax + by + a^2 + b^2 \implies p = z_x = a, q = z_y = b.$$

$$\begin{aligned} \text{L.H.S} &= z - px - qy - p^2 - q^2 \\ &= ax + by + a^2 + b^2 - ax - by - a^2 - b^2 \\ &= 0 \end{aligned}$$

Hence $z = ax + by + a^2 + b^2$ is a solution of $z - px - qy - p^2 - q^2 = 0$.

To show that $z = ax + by + a^2 + b^2$ is a complete integral of $z - px - qy - p^2 - q^2 = 0$.

$F(x, y; a, b) = ax + by + a^2 + b^2$

$$\begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix} = \begin{pmatrix} x + 2a & 1 & 0 \\ y + 2b & 0 & 1 \end{pmatrix}$$

Rank of the above matrix is 2.

Hence $z = ax + by + a^2 + b^2$ is a complete integral of $z - px - qy - p^2 - q^2 = 0$.

Unit I : Envelope of one parameter family of surfaces

- Let S_a be a family of one parameter surfaces $z = F(x, y; a)$ where a is the parameter. Consider the following system of equations.

$$z = F(x, y; a),$$
$$0 = F_a(x, y; a).$$

  The **envelope** E of S_a, if exists, is defined as the set of all $(x, y, z) \in \mathbb{R}^3$ satisfying the above system of equations for some value of the parameter a.
- For a fixed value of a, these two equations determine a curve C_a. The envelope E of the family of surfaces S_a is the union of all these curves C_a.
- The envelope E of the family of surfaces S_a, is obtained by eliminating a between

$$z = F(x, y; a), \tag{2}$$

$$0 = F_a(x, y; a). \tag{3}$$

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Lemma 0.4

Consider the partial differential equation $f(x, y, z, p, q) = 0$

Let S_a be a one parameter family of solutions $z = F(x, y; a)$ where a is the parameter of $(*)$.

Then the envelope of this family, if it exists, is also a solution of

$$f(x, y, z, p, q) = 0.$$

Unit I : Envelope of one parameter family of surfaces

- Let $S_{a,b}$ be a family of surfaces of two parameters a and b given by $z = F(x, y; a, b)$

  Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be any function.

  Let $S_{a,\phi}$ be the one-parameter family of surfaces given by $z = F\Big(x, y, a, \phi(a)\Big)$.

  Consider the following system of equations.

$$\begin{aligned} z &= F(x, y; a, \phi(a)), \\ 0 &= F_a + F_b\phi'(a). \end{aligned}$$

  The envelope of $S_{a,\phi}$, if exists, is defined as the set of all $(x, y, z) \in \mathbb{R}^3$ satisfying the above system of equations for some value of the parameter a.

- Let $S_{a,b}$ be a two parameter family $z = F(x, y, a, b)$ of complete solutions of $f(x, y, p, q) = 0$ where a, b are the parameters. Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be any function.

  Let $S_{a,\phi}$ be a family of the surfaces $z = F(x, y; a, \phi(a))$.

  Then the envelope of $S_{a,\phi}$ is also a solution of $f(x, y, p, q) = 0$.

  This solution is called a **General integral** of $f(x, y, z, p, q) = 0$.
- When a particular function ϕ is used, we obtain a **particular integral** of the partial differential equation.

  Different choices of ϕ may give different particular solutions of the partial differential equation.

Example 0.5

Consider $f(x,y,z,p,q) = z - px - qy - p^2 - q^2 = 0$.

Given that $z = F(x,y;a,b) = ax + by + a^2 + b^2$ is a complete integral.

If $b = \sqrt{1-a^2}$ then find the particular integral.

The envelope E of the family $z = F(x,y,a,\phi(a))$ is obtained by eliminating a between $z = F(x,y,a,\phi(a))$ and

$$F_a(x,y,a,b) + F_b(x,y,a,b)\phi'(a) = 0$$

$b = \sqrt{1-a^2} \implies \phi(a) = \sqrt{1-a^2}$. So $\phi'(a) = \dfrac{-a}{\sqrt{1-a^2}}$

$$z = F(x,y;a,b) = ax + by + a^2 + b^2$$

$$F_a + F_b * \phi'(a) = 0 \implies (x+2a) + (y+2b) * \frac{-a}{\sqrt{1-a^2}} = 0$$

Hence $a = \dfrac{x}{\sqrt{x^2+y^2}}$. Putting this in $z = ax + by + a^2 + b^2$, we get,

$$z = \frac{x^2}{\sqrt{x^2+y^2}} + \sqrt{\frac{y^2}{x^2+y^2}} * y + 1$$

$$z = \frac{x^2}{\sqrt{x^2+y^2}} + \frac{y^2}{\sqrt{x^2+y^2}} + 1$$

$$= \sqrt{x^2+y^2} + 1$$

Hence the envelope is given by $z = \sqrt{x^2+y^2} + 1$.

The particular integral is $z = \sqrt{x^2+y^2} + 1$.

- Let $S_{a,b}$ be a family of two parameter surfaces $z = f(x, y; a, b)$ where a, b are the parameters. Consider the following system of equations.

$$\begin{aligned} z &= F(x, y; a, b), \\ 0 &= F_a(x, y; a, b), \\ 0 &= F_b(x, y; a, b). \end{aligned}$$

The envelope E of $S_{a,b}$, if exists, is defined as the set of all $(x, y, z) \in \mathbb{R}^3$ satisfying the above system of equations for some values of the parameters a and b.

Lemma 0.6

Let $S_{a,b}$ be a two parameter family of complete integrals $z = F(x, y; a, b)$ of $f(x, y, z, p, q) = 0$ where a, b are the parameters. Then the envelope of $S_{a,b}$ is also a solution of $f(x, y, p, q) = 0$.

(This is Lemma no. 1.3.2 in our syllabus)

- This solution is called a **singular** integral of $f(x, y, z, p, q) = 0$.

Example 0.7

Consider $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0$.

Given that $z = F(x, y; a, b) = ax + by + a^2 + b^2$ is a complete integral.

Find the singular integral.

Unit I : Example: Finding Singular Integral

- For singular integral, we take $z = F(x, y; a, b)$ and the we eliminate a and b using the equations $F_a(x, y; a, b) = 0$ and $F_b(x, y; a, b) = 0$. Here,

$$\begin{aligned} z = F(x, y; a, b) &= ax + by + a^2 + b^2 \\ F_a &= x + 2a \\ F_b &= y + 2b \\ F_a = 0, F_b = 0 \implies \quad & a = -\frac{x}{2}, \quad b = -\frac{y}{2} \end{aligned}$$

Substituting these values in $z = F(x, y; a, b) = ax + by + a^2 + b^2$, we get

$$z = -\frac{x^2 + y^2}{4}$$

That is, $\mathbf{4z = -(x^2 + y^2)}$

Unit I : Another method to Find Singular Integral

Lemma 0.8

Let $z = F(x, y; a, b)$ be a complete integral of $f(x, y, z, p, q) = 0$ and $z = F(x, y, a(x, y), b(x, y))$ be the singular integral of $f(x, y, z, p, q) = 0$. Then the singular integral satisfies the equations

$$\begin{aligned} f(x, y, z, p, q) &= 0, \\ f_p(x, y, z, p, q) &= 0, \\ f_q(x, y, z, p, q) &= 0. \end{aligned}$$

Example 0.9

Consider $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0$.

Given that $z = F(x, y; a, b) = ax + by + a^2 + b^2$ is a complete integral.

Find the singular integral using the above lemma.

We know that singular integral satisfies:

$$\begin{aligned} f(x, y, z, p, q) = 0 \quad &\implies z - px - qy - p^2 - q^2 = 0, \\ f_p(x, y, z, p, q) = 0 \quad &\implies x - 2p = 0, \\ f_q(x, y, z, p, q) = 0 \quad &\implies -y - 2q = 0. \end{aligned}$$

This implies $p = -\dfrac{x}{2}, q = -\dfrac{y}{2}$.

Hence the singular solution is $\boldsymbol{z = -\dfrac{x^2 + y^2}{4}}$

The Cauchy Problem

- Given a first order partial differential equation and a curve in space, the Cauchy problem is to find an integral surface (i.e. a solution) of the given partial differential equation which contains the given curve.

  In other words, given a partial differential equation (not necessarily non-linear)

$$f(x, y, z, p, q) = 0$$

and a curve $x = x(s), y = y(s), z = z(s), s \in [a, b]$,

the Cauchy problem is to find a solution $z = z(x, y)$ of the pde

such that $z(s) = z(x(s), y(s))$ for all $s \in [a, b]$.
(Note: We will be studying this in unit II in detail.)

Theorem 0.10

The general solution of the Lagrange equation

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z), \qquad (*)$$

where P, Q and R are continuously differentiable functions on the domain $D \subseteq \mathbb{R}^3$ is $\phi(u,v) = 0$ where ϕ is an arbitrary function and

$$u(x,y,z) = c_1 \quad \text{and} \quad v(x,y,z) = c_2$$

are two independent solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.
(Lagrange's auxiliary equations of $(*)$)

The general solution (or integral) of (1) is written in one of the following three equivalent forms:

$$\phi(u,v) = 0, \quad u = G(v) \quad \text{or} \quad v = H(u)$$

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Theorem 0.11

A general solution of the quasi-linear partial differential equation

$$P_1\frac{\partial z}{\partial x_1}+P_2\frac{\partial z}{\partial x_2}+\cdots+P_n\frac{\partial z}{\partial x_n}=R.$$

where $P_1, P_2, \ldots, P_n, R$ are continuously differentiable functions of $x_1, x_2, \ldots, x_n$ and z, not simultaneously zero,
is the relation $\phi(u_1, u_2, \ldots, u_n) = 0$ where ϕ is an arbitrary differentiable function and $u_1(x_1, x_2, \ldots, x_n, z) = c_1, u_2(x_1, x_2, \ldots, x_n, z) = c_2, \ldots, u_n(x_1, x_2, \ldots, x_n, z) = c_n$ are independent solutions of the equations

$$\frac{dx_1}{P_1}=\frac{dx_2}{P_2}=\cdots=\frac{dx_n}{P_n}=\frac{dz}{R}.$$

Unit I :Type 1: Solving Lagarange's Equation

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Solve the pde of $\frac{y^2 z}{x} p + xz\, q = y^2$

Lagrange's auxiliary eqns are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \qquad (1).$$

Taking the first two fractions, from (1)

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz}.$$

$$x^2\ dx = y^2\ dy$$

$$x^3 = y^3 + c_1 \qquad (2)$$

Taking the first and the last from (1).

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dz}{y^2}.$$

$$x\ dx = z\ dz.$$

$$x^2 = z^2 + c_2 \qquad (3).$$

From (2) and (3), the required general solution is
$\boldsymbol{\phi(x^3 - y^3, x^2 - z^2) = 0.}$
Another form of the general integral is $\boldsymbol{G(x^3 - y^3) = x^2 - z^2.}$

Unit I :Type 2: Solving Lagarange's Equation

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Solve the pde $p + 3\ q = 5z + \tan(y - 3x)$

Lagrange's auxiliary eqns are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{\tan(y-3x)} \qquad (1).$$

Taking the first two fractions, from (1)

$$\frac{dx}{1} = \frac{dy}{3}.$$

$$y - 3x = c_1 \qquad (2).$$

c_1 denotes a constant.

Taking the first and the last from (1).

$$\frac{dx}{1} = \frac{dz}{5z + \tan c_1}.$$

$$x = \frac{1}{5}\ln(5z + \tan c_1) = c_2$$

$$5x - \ln(5z + \tan c_1) = c_2 \qquad (3).$$

(c_2 denotes a constant)

From (2) and (3), the required general solution is

$$\boldsymbol{\phi\,(y - 3x, 5x - \ln(5z + \tan c_1)) = 0}$$

where ϕ is an arbitrary function.

Unit I :Type 3: Solving Lagarange's Equation

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Let P_1, Q_1 and R_1 be functions of x, y and z.

Then each fraction in Lagrange's auxiliary eqns
$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ is equal to

$$\frac{P_1\ dx + Q_1\ dy + R_1\ dz}{P_1P + Q_1Q + R_1R} \qquad (*)$$

If $P_1P + Q_1Q + R_1R = 0$, then the numerator of $(*)$ is also 0. This gives

$$P_1\ dx + Q_1\ dy + R_1\ dz = 0 \qquad (**)$$

This can be integrated to get $u_1(x, y, z) = c_1$. This method may be repeated to get another integral $u_2(x, y, z) = c_2$. P_1, Q_1, R_1 are called multiplayers.

Solve the pde $\quad yz\frac{b-c}{a}p + zx\frac{c-a}{b}q = \frac{a-b}{c}xy.$

Lagrange's auxiliary eqns are

$$\frac{a\ dx}{yz(b-c)} = \frac{b\ dy}{zx(c-a)} = \frac{c\ dz}{xy(a-b)} \quad \cdots (1).$$

Unit I :Type 3: Solving Lagarange's Equation

Choosing x, y, z as multipliers, each fraction equals

$$= \frac{ax\ dx + by\ dy + cz\ dz}{0}.$$

$$\implies ax\ dx + by\ dy + cz\ dz = 0.$$

Integrating

$$a\frac{x^2}{2} + b\frac{y^2}{2} + c\frac{z^2}{2} = c_1$$

$$\implies ax^2 + by^2 + cz^2 = c_1 \qquad \cdots (2)$$

(c_1 being arbitrary constant).

Now, choosing ax, by and cz as multipliers for eqn (1), we get

$$= \frac{a^2x\ dx + b^2y\ dy + c^2z\ dz}{xyz\Big(a(b-c) + b(c-a) + c(a-b)\Big)}$$

$$= \frac{a^2x\ dx + b^2y\ dy + c^2z\ dz}{0}$$

$$\implies a^2x\ dx + b^2y\ dy + c^2z\ dz = 0$$

Integrating,

$$a^2x^2 + b^2y^2 + c^2z^2 = c_2 \quad \cdots (3)$$

From (2) and (3), the required general solution is

$$\phi\left(ax^2 + by^2 + cz^2, a^2x^2 + b^2y^2 + c^2z^2\right) = 0.$$

where ϕ is an arbitrary function.

Unit I :Type 4: Solving Lagarange's Equation

Let P_1, Q_1 and R_1 be functions of x, y and z.
Then all fractions in $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ are equal to

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R} \qquad (**).$$

Suppose the numerator is the exact differential of the denominator of $(**)$.
Then $(**)$ can be combined with a suitable fraction in $(*)$ to give an integral.

Unit I :Type 4: Solving Lagarange's Equation

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

$(y+z)\,p+(z+x)\,q=x+y$

Lagrange's auxiliary eqns are

$$\frac{dx}{(y+z)}=\frac{dy}{(z+x)}=\frac{dz}{x+y}\quad\cdots(1).$$

Choosing $1,-1,0$ as multipliers, each fraction equals

$$=\frac{dx-dy}{y-x}=-\frac{d(x-y)}{x-y}\qquad\cdots(2).$$

Choosing $0,1,-1$ as multipliers, each fraction equals

$$=\frac{dy-dz}{z-y}=-\frac{d(y-z)}{y-z}\qquad\cdots(3).$$

Choosing $1,1,1$ as multipliers, each fraction equals

$$=\frac{dx+dy+dz}{y+z+z+x+x+y}=\frac{dx+dz}{2(x+y+z)}\qquad\cdots(4).$$

From $(2),(3)$ and (4), we have,

$$-\frac{d(x-y)}{x-y}=-\frac{d(y-z)}{y-z}=\frac{dx+dy+dz}{2(x+y+z)}\qquad\cdots(5).$$

Taking the first two fractions of (5)

$$\frac{d(x-y)}{x-y}=\frac{d(y-z)}{y-z}$$

Integrating, $\Longrightarrow \dfrac{x-y}{y-z} = c_1 \qquad \cdots(6).$

Taking the first and third fractions of (5)

$$-\frac{d(x-y)}{x-y} = \frac{dx+dy+dz}{2(x+y+z)}$$

Integrating, $-2\ln(x-y) = \ln(x+y+z) + \ln C_2$

$$\ln(x+y+z) + 2\ln(x-y) = -\ln C_2$$

$$\ln(x+y+z)(x-y)^2 = \ln c_2$$

$$(x+y+z)(x-y)^2 = c_2 \qquad (7).$$

From (6) and (7), the required general solution is

$$\phi\left(\frac{x-y}{y-z}, (x+y+z)(x-y)^2\right) = 0.$$

where ϕ is an arbitrary function.

Unit I : Pfaffian Differential Equation

- By a **Pfaffian differential equation**, we mean a differential equation of the form

$$F_1(x_1, x_2, \ldots, x_n)dx_1 + \cdots + F_n(x_1, x_2, \ldots, x_n)dx_n = 0 \quad (*)$$

where $F_i's, 1 \leq i \leq n$ are continuous functions.

The expression on the LHS is called a **Pfaffian differential form**.

- A Pfaffian differential form

$$F_1(x_1, x_2, \ldots, x_n)dx_1 + \cdots + F_n(x_1, x_2, \ldots, x_n)dx_n$$

is said to be **exact** if we can find a **continuously differentiable function $u(x_1, x_2, \ldots, x_n)$** such that

$$du = F_1(x_1, x_2, \ldots, x_n)dx_1 + \cdots + F_n(x_1, x_2, \ldots, x_n)dx_n.$$

- A **Pfaffian differential equation**
  $$F_1(x_1, x_2, \dots, x_n)dx_1 + \cdots + F_n(x_1, x_2, \dots, x_n)dx_n = 0 \qquad (**)$$
  is said to be **exact** if the Pfaffian differential form on the LHS of the equation is exact.
- That is, the Pfaffian differential equation $(**)$ is said to be exact if we can find a **continuously differentiable function** $u(x_1, x_2, \dots, x_n)$ such that
  $$du = F_1(x_1, x_2, \dots, x_n)dx_1 + \cdots + F_n(x_1, x_2, \dots, x_n)dx_n.$$
- The function $u(x_1, x_2, \dots, x_n) = c$, is called the **integral** of the corresponding **Pfaffian differential equation**.
- The Pfaffian differential equation $(**)$ is said to be **integrable** if there exists a **non-zero differentiable function** $\mu(x_1, x_2, \dots, x_n)$ such that the Pfaffian differential form
  $$\mu\left(F_1(x_1, x_2, \dots, x_n)dx_1 + \cdots + F_n(x_1, x_2, \dots, x_n)dx_n\right)$$
  is exact.
- The function $\mu(x_1, x_2, \dots, x_n)$ is called an **integrating factor** of $(**)$.

Theorem 0.12

There always exists an **integrating factor** for a Pfaffian differential equation in two variables $(P(x,y)\ dx + Q(x,y)\ dy = 0)$.

Lemma 0.13

Let $u(x,y) = c_1$ and $v(x,y) = c_2$ be two functions of x and y such that

$$\frac{\partial v}{\partial y} \neq 0.$$

If, further

$$\frac{\partial(u,v)}{\partial(x,y)} = 0,$$

then there exists a relation $F(u,v) = 0$

between u and v **not involving x and y explicitly.**

Recall: If $\overline{\mathbf{X}} = (P, Q, R)$ then the curl of $\overline{\mathbf{X}}$ is defined by

$$\text{curl } \overline{\mathbf{X}} = (R_y - Q_z)\hat{\mathbf{i}} + (P_z - R_x)\hat{\mathbf{j}} + (Q_x - P_y)\hat{\mathbf{k}}$$

- The definition of curl can be difficult to remember. To help with remembering, we use the following determinant formula.

$$\text{curl } \overline{\mathbf{X}} = \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}$$

Lemma 0.14

If $\overline{\mathbf{X}} = \Big(P(x,y,z), Q(x,y,z), R(x,y,z)\Big)$ and μ is an arbitrary nonzero differentiable function of x, y and z then

$$\overline{\mathbf{X}} \cdot \ \text{curl } \overline{\mathbf{X}} = 0 \quad \text{if and only if} \quad \mu\overline{\mathbf{X}} \cdot \ \text{curl } \Big(\mu\overline{\mathbf{X}}\Big) = 0.$$

Theorem 0.15

A necessary and sufficient condition that the Pfaffian differential equation

$$\overline{\mathbf{X}} \cdot \overline{\mathbf{dr}} = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0 \tag{4}$$

to be integrable is that $\overline{\mathbf{X}} \cdot \ \text{curl } \overline{\mathbf{X}} = 0$

Unit I : Condition for Pfaffian Differential Equation to be exact

Remark 0.16

Necessary and sufficient condition for the Pfaffian differential equation $\overline{\mathbf{X}} \cdot \overline{\mathbf{dr}} = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0$ to be **exact** is

$$\text{curl } \overline{\mathbf{X}} = \overline{\mathbf{0}}$$

That is, $R_y - Q_z = 0, P_z - R_x, Q_x - P_y = 0$

Example 0.17

Show that the following Pfaffian differential equation is exact and find its integral. $y\,dx + x\,dy + 2z\,dz = 0$.

Here $P = y, Q = x, R = 2z$.

$$\text{curl } \overline{\mathbf{X}} = \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 2z \end{pmatrix} = (0,0,0).$$

Clearly, $y\,dx + x\,dy + 2z\,dz = d(xy + z^2)$

So, $d(xy + z^2) = 0$. This implies $d(xy + z^2) = c$.

Hence the integral is $u(x, y, z) = xy + z^2 = c$.

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Example 0.18

Find the integral of $yz\ dx + 2xz\ dy - 3xy\ dz = 0$

Here $P = yz, Q = 2xz, R = -3xy$.

$$\text{curl } \overline{\mathbf{X}} = \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xz & -3xy \end{pmatrix} = \hat{\mathbf{i}}(-3x-2x) - \hat{\mathbf{j}}(-3y-y) + \hat{\mathbf{k}}(2z-z).$$

$$\text{curl } \overline{\mathbf{X}} = -5x\ \hat{\mathbf{i}} + 4y\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}$$

$$\overline{\mathbf{X}} \cdot \text{curl } \overline{\mathbf{X}} = -5xyz\ + 8xyz\ - 3xyz = 0$$

Hence given equation is integrable.

Keep z as a constant and write the differential equation as follows:

$$yz\ dx + 2xz\ dy = 0$$

We find the solution of the above equation.

$\frac{dx}{x} = -2\frac{dy}{y}$

$x * y^2 = c_1$ whee c_1 is a constant and it may contain z.

So, $U(x, y, z) = xy^2 = c_1$.

Now we find Integrating factor μ

Consider equation $\frac{\partial U}{\partial x} = \mu * P$ OR $\frac{\partial U}{\partial y} = \mu * Q$

Here $y^2 = \mu * yz \implies \mu = \frac{y}{z}$.

Unit I : Example 2 of Pfaffian differential Equation

My Inspiration
Late. Shivlal Dhamone
and
Shri. V. G. Patil Saheb

Subject Teacher
Santosh Dhamone

Now we find $K = \left(\mu * R - \frac{\partial U}{\partial z}\right)$

$K = \frac{y}{z} * (-3xy) - 0 = -\frac{3xy^2}{z} = -\frac{3U}{z}.$

Substitute in the equation $\frac{dU}{dz} + K = 0$

This implies $\frac{dU}{dz} - \frac{3U}{z} = 0$

We solve this equation.

Solution is $U = cz^3$.

This means $xy^2 = cz^3$.

Therefore the integral of the given Pfaffian equation is

$$u(x, y, z) = \frac{xy^2}{z^3} = c.$$