Duration: 2 ½ Hrs

Marks: 75

N.B. : (1) All questions are compulsory.
(2) Figures to the right indicate full marks.

1. (a) Attempt any One of the following: (8)
   (i) Let $z = f(x, y)$ be a homogeneous function of x and y of degree n. Then prove that the function $f(x, y)$ satisfies the first order partial differential equation $x\,p + y\,q = nz$.
   (ii) Let $S_{a,b}$ be a two parameter family of solutions $z = F(x, y; a, b)$ of the partial differential equation $f(x, y, z, p, q) = 0$ where a, b are the parameters. Then prove that the envelope of this family is also a solution of $f(x, y, z, p, q) = 0$.

   (b) Attempt any Two of the following. (12)
   (i) For the Lagrange's equation $y^2(x-y)\,p+x^2(y-x)\,q = z(x^2+y^2)$, state the auxiliary equations. Also show that each ratio in the auxiliary equations equals the fraction $\dfrac{dx - dy}{(x-y)(x^2+y^2)}$ and hence solve the given partial differential equation.
   (ii) Solve the integrable Pfaffian differential equation
   $$(y^2 + yz)\,dx + (xz + z^2)\,dy + (y^2 - xy)\,dz = 0.$$
   (iii) (A) When is a partial differential equation $f(x, y, z, p, q) = 0$ called a first order semi-linear partial differential equation? Give an example of a first order semi-linear partial differential equation.
   (B) When is a partial differential equation $f(x, y, z, p, q) = 0$ called a first order quasi-linear partial differential equation? Give an example of a first order quasi-linear partial differential equation.

2. (a) Attempt any One of the following. (8)
   (i) Define a Compatible system of two first order Partial differential equations. Prove that the partial differential equations $p = M(x, y)$ and $q = N(x, y)$ are compatible if and only if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.
   (ii) If the given partial differential equation is in the form $h(x, p) = k(y, q)$, then show that the equation has a complete integral of the form $z = \int P(a, x)\,dx + \int Q(a, y)\,dy + b$ where a and b are arbitrary constants and P, Q are functions.

   (b) Attempt any Two of the following. (12)
   (i) Show that the equations $xp = yq$ and $z(xp + yq) = 2xy$ are compatible. Also find a common solution to them.
   (ii) The partial differential equation $p(1+q) = qz$ is of the form $f(z, p, q) = 0$. Find its complete integral.
   (iii) Write the auxiliary equations of Jacobi's method for the equation
   $$z^2 + zu_z - u_x^2 - u_y^2 = 0.$$

3. (a) Attempt any One of the following. (8)

(i) Describe the steps followed in finding the integral surface for the partial differential equation $P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z)$ containing the data curve $x = x_0(s), y = y_0(s), z = z_0(s), s \in I$, where $I \subseteq \mathbb{R}$ is an interval and x_0, y_0, z_0 are continuously differentiable functions on I.

(ii) What is Monge cone at (x_0, y_0, z_0) for the partial differential equation $f(x, y, z, p, q) = 0$? Show that the Monge cone is generated by the straight lines given by $\dfrac{x - x_0}{f_p} = \dfrac{y - y_0}{f_q} = \dfrac{z - z_0}{p f_p + q f_q}$.

(b) Attempt any Two of the following. (12)

(i) Show that the characteristic curves of $z\, p + q = 0$ containing the initial data curve $C : x_0(s) = s, y = y_0(s) = 0, z_0(s) = f(s)$ where

$$f(s) = \begin{cases} 1 \text{ if } s \leq 0, \\ 1 - s \text{ if } 0 \leq s \leq 1, \\ 0 \text{ if } s \geq 1 \end{cases}$$

are straight lines given by $x = y\, f(s) + s$.

(ii) Find the solution of the initial value problem for the quasi-linear equation $p - z\, q = -z$ for all y and $x > 0$ for the initial data curve $C : x_0(s) = 0, y_0(s) = s, z_0(s) = -2s, \quad -\infty < s < \infty$.

(iii) Find the initial strip and the characteristic strips of the equation $xp + yq - pq = 0$ with the initial data curve $C : x_0(s) = 2s, y_0(s) = 0, z_0(s) = s$.

4. Attempt any Three of the following. (15)

(a) Eliminate the arbitrary function ϕ from $\phi(z - xy, x^2 + y^2) = 0$ and find the corresponding partial differential equation.

(b) Solve the Lagrange's differential equation $xz\, p + yz\, q = xy$.

(c) If $f(x, y, z, p, q) = x\, p - y\, q - x = 0$ and $g(x, y, z, p, q) = x^2\, p + q - xz = 0$, find $\dfrac{\partial(f, g)}{\partial(x, p)} + p\, \dfrac{\partial(f, g)}{\partial(z, p)} + \dfrac{\partial(f, g)}{\partial(y, q)} + q\, \dfrac{\partial(f, g)}{\partial(z, q)}$.

(d) Find a complete integral of partial differential equation $z = px + qy + pq$ using Clairaut Equation method.

(e) Find the characteristic differential equations for the partial differential equation

$$p^2 x + qy - z = 0.$$

(f) Find the characteristic curves $p + q = z^2$ through the initial data curve

$$x = x_0(s) = s, y = y_0(s) = 0, z = z_0(s) = f(s), s \in I$$

where I is an interval and $f : I \longrightarrow \mathbb{R}$.

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