Chapter 2

Second Order Linear Differential Equation

In This Chapter:

2.1 INTRODUCTION:

In the previous chapter we have studied first order linear differential equations. In this chapter, we consider second order linear differential equation, homogeneous and non-homogeneous equations. Different methods of obtaining general solutions to these types of equations.

These types of equations play an important role in the study of the motion of a spring. These equations describe simple harmonic motion, oscillations of mass spring system, RLC-Circuit, oscillations of a simple pendulum, etc.

We also consider the homogeneous differential equations with constant coefficients. We also present method of undetermined coefficients and method of variation of parameters for non-homogeneous equations.

Existence and uniqueness theorem is the tool which makes it possible for us to conclude that there exists only one solution to second order differential equations. We are going to state this theorem and use to solve the problems.

Let us start with the general form of second order linear differential equation.

2.2 SECOND ORDER LINEAR DIFFERENTIAL EQUATION:

The general form of the second order linear differential equation is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ ..... (1)

where P, Q and R are functions of x or constants.

This differential equation can also be written as $y'' + Py' + Qy = R$.

We write the Initial Value Problem (IVP) as follows:

$y'' + Py' + Qy = R, \qquad y(x_0) = y_0,\ y'(x_0) = y'_0$

Differential Operator:

Let D be the symbol which denotes differentiation w.r.t. x.

$$\therefore Dy = \frac{dy}{dx},\ D^2y = D(Dy) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$\therefore \frac{1}{D}$ means integration

$\therefore \frac{1}{D^2}$ means integration twice

i.e. $\frac{1}{D}(x^2) = \frac{x^3}{3}$

and $\frac{1}{D^2}(x^2) = \frac{1}{D}\left[\frac{1}{D}(x^2)\right] = \frac{1}{D}\left[\frac{x^3}{3}\right] = \frac{x^4}{12}$

$\therefore$ Equation (1) can be written in operator form as

$D^2y + PDy + Qy = R$

Or

$(D^2 + PD + Q)y = R$

There are two types of equations depending upon the value of R.

(1) **Homogeneous:**

If $R = 0$, the equation is called homogeneous.

(2) **Non-homogeneous:**

If $R \neq 0$, the equation is called non-homogeneous.

e.g.

(i) $y'' + \sqrt{\frac{m}{L}} \sin y = 0$ is a non-linear second order differential equation.

(ii) $y'' = y$ is linear homogeneous second order differential eqution.

(iii) $y'' + y' + y = \sin x$ linear non-homogeneous second order differential, equation.

(iv) $ax^2y'' + bxy' + cy = 0$, $a, b, c \in R$ & $c \neq 0$ is linear homogeneous second order differential equation. This equation is called **Euler equation** of order 2.

2.3 VECTOR SPACE:

2.3.1 Definition:

Solution Space:

The set of solutions of a differential equation is called the solution space. E.g. all solutions of the differential equation

$y'' + Py' + Qy = 0$ ..... (2)

form a solution space.

Note that $y(x) = 0$ is also a solution.

∴ Solution set of equation (2) is non-empty.

2.3.2 Definition:

Vector Space:

The solution space of a linear homogeneous differential equation is a vector space.

That is, the set of solutions of equation (2) form a vector space under addition and scaling.

Moreover this is a two dimensional real vector space.

∴ The solution sets of equations of the type $y'' - 2y' - y = 0$ where $P = -2$, $Q = -1$ are constants or $y'' + (x^2 - x)y' - (\sin x)y = 0$ where $P = (x^2 - x)$, $Q = -\sin x$ are functions of x from the vector space.

2.3.3 Remark:

Solutions of a non-homogeneous differential equation do not form a vector space.

e.g. the sum of two solutions to non-homogeneous differential equations is not a solution in general.

Let y_1 and y_2 are two solutions of equation (2) then $y_1 + y_2$, k_1y_1 or k_2y_2, $k_1y_1 + k_2y_2$ are also solutions of equation (2) which is proved by the following theorem.

2.3.4 Theorem:

If $y_1(x)$ and $y_2(x)$ are any two solutions of the differential equation $y'' + Py' + Qy = 0$. ... (1)

Then $c_1\ y_1(r) + c_2\ y_2(x)$ is also a solution, for any constants c_1 and c_2.

Proof:

Let $y(x) = c_1\ y_1(x) + c_2\ y_2(x)$

Differentiating w.r.t x, twice we get

$$y' = c_1 y_1' + c_2 y_2' \quad .....(3)$$

$$y'' = c_1 y_1'' + c_2 y_2'' \quad .....(4)$$

Substitute the values of y, y′ and y″ from (2), (3) and (4) in (1) we get

$$(c_1 y_1'' + c_2 y_2'') + P(c_1 y_1' + (2y_2') + Q(c_1 y_1 + c_2 y_2) = 0$$

Taking c_1 and c_2 common, we have

$$c_1(y_1'' + Py_1' + Qy_1) + c_2(y_2'' + Py'_2 + Qy_2) = 0$$

By assumptions y_1 and y_2 are solutions of (1)

$\therefore\ y_1'' + Py_1' + Qy_1 = 0$ and $y_2'' + Py_2' + Qy_2 = 0$

$\therefore$ Equation (1) is satisfied by equation (2).

$\therefore$ Equation (2) i.e. $y = c_1 y_1 + c_2 y_2$ is general solution of Equation (1).

i.e. $y'' + Py' + Qy = 0$

2.3.5 Theorem:

The space of solution of the second order linear differential equation $y'' + py' + Qy = 0$ forms a real vector space.

Proof:

Let V denote the solution space. Let $y_1(x), y_2(x) \in V$ from 2.3.1 taking $c_1 = c_2 = 1$ we get $y_1 + y_2 \in V$ and $c_1\ y_1 \in V$ for $c_1 \in \mathbb{R}$. Hence closure property of addition and scalar multiplication is satisfied. It can also be seen that commulative and associative property are also satisfied. $y(x) = 0$ is a solution to (1) which give the additive identity of V and for $y_1 \in V, -y_1 \in V$ taking $c_1 = -1$ and $c_2 = 0$ in theorem 2.3.1

The distributive properties of addition over scalar multiplication hold as

$$c(y_1 + y_2) = cy_1 + cy_2$$

and $(c_1 + c_2)\, y_1 = c_1y_1 + c_2y_1$

Further $c_1(c_2\, y_1) = (c_1\, c_2)y_1$

and $1.y_1 = y_1$ clearly holds which proves that V is a vector space.

2.4 WRONSKIAN AND LINEAR INDEPENDENCE:

2.4.1 Definition:

Wronskian:

Let $y_1(x)$ and $y_2(x)$ are solutions of the equation (2). Then define a determinant.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

W is called Wronskian determinant or Wronskian of y_1 and y_2.

2.4.2 Theorem:

Let $y_1(x)$ and $y_2(x)$ be any two solutions to the differential equation $y'' + Py' + Qy = O$ on $[a, b]$, then. Their Wronskian $W(y_1, y_2)$ is either identically zero or never zero.

Proof: $W(y_1, y_2) = y_1\, y_2' - y_2\, y_1'$

$$W' = y_1\, y_2'' + y_1'\, y_2' - y_2\, y_1'' - y_2'\, y_1'$$

$$= y_1\, y_1'' - y_2\, y_1''$$

$y_1(x)$ and $y_2(x)$ are solutions to $y'' + Py' + Qy = 0$

$$\Rightarrow \quad y_1'' + Py_1' + Q\, y_1 = 0 \qquad \dots (*)$$

$$y_2'' + P_2' + Qy_2 = 0 \qquad \dots (**)$$

(*) × y_2 gives $y_2\, y_1'' + Py_1'\, y_2 + Qy_1\, y_2 = 0$

(**) × y_1 gives $y_1\, y_2'' + Py_2'\, y_1 + Qy_1\, y_2 = 0$

Subtracting $y_2\, y_1'' - y_1\, y_2'' + P(y_1'\, y_2 - y_2'\, y_1) = 0$

$$-W' - PW = 0$$

$$\Rightarrow \quad \frac{dW}{dx} + PW = 0$$

$$\Rightarrow \quad \frac{dW}{dx} = -PW \quad \Rightarrow W = c\,e^{-\int P dx}$$

Note that $e^{-\int P dx}$ is an exponential function and for any value of p this value is never zero.

Hence $W = c\,e^{-\int P dx}$ is identically zero if $c = 0$ and if $c \neq 0$ then W is never zero.

Hence either the wronskraion is identically zero of Wronskian is never zero.

2.4.3 Definition:

Linearly Dependent:

Let $y_1, y_2: [a, b] \rightarrow \mathbb{R}$ be two real valued functions. $y_1(x)$ and $y_2(x)$ are said to be linearly dependant on [a, b] if there exists a real number α such that $y_1(x) = \alpha\, y_2(x)\ \forall\, x \in [a, b]$. y_1 and y_2 are said to be linearly independent if they are not linearly dependant on [a, b].

2.4.4 Theorem:

Let $y_1(x)$ and $y_2(x)$ be any two solutions to the differential equation $y'' + Py' + Qy = 0$ on the interval [a, b], then their Wronskian $W(y_1, y_2) = y_1 y_1' - y_2 y'$ is identically zero if and only if $y_1(x)$ and $y_2(x)$ are linearly dependant on [a, b].

Proof:

Suppose $y_1(x)$ and $y_2(x)$ are linearly dependant on [a, b]. If either of $y_1(x)$ or $y_2(x)$ is identically zero on [a, b] then clearly $W(y_1, y_2) = 0$. Hence we now assume $y_1(x) \neq 0$ and $y_2(x) \neq 0$ and y_1 and y_2 are linearly dependant on [a, b].

$\Rightarrow y_1(x) = \alpha\, y_2(x)$ for some $\alpha \in \mathbb{R}\ \forall\, x \in [a, b]$ with $\alpha \neq 0$

$\Rightarrow y_1' = \alpha\, y_2'$

and $W(y_1, y_2) = y_1 y_2' - y_2 y_1' = (\alpha y_2)\, y_2' - y_2(\alpha\, y_2') = 0$

$$= \alpha\,(y_2 y_2' - y_2 y_2') = 0$$

Conversly suppose $W(y_1, y_2)$ is identically zero. If $y_1(x)$ is identically zero, then for any $y_2(x)$, $y_1(x) = 0 = 0.y_2(x)$ for all $x \in [a, b] \Rightarrow y_1$ and y_2 are linearly dependant.

Now we assume that $y_1(x)$ is not identically zero on [a, b] and y_1 is continuous on [a, b]. Hence there exists an interval $[c, d] \subseteq [a, b]$ in which $y_1(x) \neq 0 \quad \forall x \in [c, d]$

$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ is identically zero on [a, b]

Since $y_1(x) \neq 0 \ \forall x \in [c, d]$ we divide $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ by $y_1^2(x) \ \forall x \in [c, d]$.

$$\Rightarrow \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = 0 \quad \forall \quad x \in [c, d]$$

$$\Rightarrow \left(\frac{y_2}{y_1}\right)' = 0 \quad \forall \quad x \in [c, d]$$

Integrating $\frac{y_2}{y_1} = k$ where k is a constant

$\Rightarrow y_2(x) = k\, y_1(x) \quad \forall \quad x \in [c, d]$

Thus the functions y_2 and ky_1 have equal values in [c, d] and value of their derivatives in the interval are also equal as

$$y_2'(x) = k\, y_1'(x)$$
$$= (k\, y_1)'(x)$$

Thus y_2 and $k\, y_1$ have equal values in $[c, d] \subseteq [a, b]$ and equal derivatives in $[c, d] \subseteq [a, b]$. By Existence and Uniqueness theorem of solution of second order homogeneous linear differential equation.

$$y_2(x) = k\, y_1(x) \quad \forall x \in [a, b]$$

$\Rightarrow y_1$ and y_2 are linearly dependant on [a, b].

2.4.5 Remark:

The above theorem indicates that the Wrongkian $W(y_1, y_2)$ is never zero if and only if $y_1(x)$ and $y_2(x)$ are linearly independent.

Illustration 1:

$y_1 = \cos x, \quad y_2 = \sin x, \quad x \in \mathbb{R}$

Solution:

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$\therefore W = 1 \neq 0$

$\therefore y_1$ and y_2 are linearly independent.

Illustration 2:

$y_1 = e^{ax}, \quad y_2 = e^{-ax}$

Solution:

$$W(y_1, y_2) = \begin{vmatrix} e^{ax} & e^{-ax} \\ ae^{ax} & -ae^{-ax} \end{vmatrix} = -a-a = -2a$$

If $a = 0$ then y_1 and y_2 are **linearly dependent.**

If $a \neq 0$ then y_1 and y_2 are **linearly independent.**

Illustration 3:

$y_1 = x^2, \quad y_2 = x^2 \log x, \quad x \neq 0$

Solution:

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^2 \log x \\ 2x & x + 2x\log x \end{vmatrix}$$

$\therefore W(y_1, y_2) = (x^3 + 2x^3 \log x) - (2x^3 \log x)$

$\therefore W(y_1, y_2) = x^3 \neq 0 \qquad \because x \neq 0$

$\therefore y_1$ and y_2 are linearly independent.

Illustration 4:

$y_1 = x^2|x|, \quad y_2 = x^3, \quad x \in \mathbb{R}$

Solution:

$y_1 = x^2(-x)$ if $x < 0$

$\quad = x^2(x)$ if $x \geqslant 0$

For $x < 0$

$$W(y_1, y_2) = \begin{vmatrix} -x^3 & x^3 \\ -3x^2 & 3x^2 \end{vmatrix} = -3x^5 + 3x^5 = 0$$

For $x \geq 0$

$$W(y_1, y_2) = \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix} = 3x^5 - 3x^5 = 0$$

$\therefore W(x_1, y_2) = 0$ for $x \in \mathbb{R}$

$\therefore y_1$ and y_2 are linearly dependent.

Illustration 5:

$y_1 = \cos 2\pi x, \quad y_2 = \sin 2\pi x$

Solution:

$$W(y_1, y_2) = \begin{vmatrix} \cos 2\pi x & \sin 2\pi x \\ -2\pi \sin 2\pi x & 2\pi \cos 2\pi x \end{vmatrix}$$

$$= 2\pi \cos^2 2\pi x + 2\pi \sin^2 2\pi x$$

$$W(y_1, y_2) = 2\pi (\cos^2 2\pi x + \sin^2 2\pi x) = 2\pi \neq 0$$

$\therefore y_1, \; y_2$ are linearly independent.

EXERCISE 2.1

Determine whether the following functions y_1 and y_2 are linearly dependent or independent.

(1) $y_1 = 2\sin^2 x$, $\quad y_2 = 1 - \cos^2 x$

(2) $y_1 = x_1$, $\quad y_2 = x^2$

(3) $y_1 = \cos ax$, $\quad y_2 = \sin ax$, $\quad a \neq 0$

(4) $y_1 = \log x$, $\quad y_2 = \log x^n$, $\quad n > 0$

(5) $y_1 = x^2$, $\quad y_2 = 5x^2$

(6) $y_1 = e^x$, $\quad y_2 = xe^x$

ANSWER 2.1

(1), (4) & (5) linearly dependent.

(2), (3) & (6) linearly independent.

2.5 GENEARL SOLUTION OF HOMOGENEOUS DIFFERENTIAL EQAUTION:

2.5.1 Theorem:

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogeneous differential equaltion $y'' + P(x)\,y' + Q(x)\,y = 0 \;\cdots (*)$

on the interval [a, b], then $c_1 y_1(x) + c_2 y_2(x)$ is the general solution of (*).

Proof:

Let y(x) be any solution to (*) we will show that the constants c_1 and c_2 can be found so that $y(x) = c_1y_1(x) + c_2y_2(x) \quad \forall x \in [a, b]$.

By Existence and Unique theorem of solution of linear homogeneous differential equation if $x_0 \in [a, b]$ any point and y_0 and y_0' are any two real numbers then (*) has a unique solution y(x) such that $y(x_0) = y_0$ and $y_0'(x_0) = y_0$.

In otherwords, a solution of (*) is completely determined by its value at a point and the value of its derivative at the same point.

$y_1(x)$ and $y_2(x)$ are 2 linearly independent solutions of (*). By theorem 2.3.1, $c_1y_1(x) + c_2y_2(x)$ is also a solution of (*).

Since $y_1(x)$ and $y_2(x)$ are linearly independent by theorem 2.4.4 their Wrongkian $W(y_1, y_2)$ is never zero (i.e)

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' \neq 0 \qquad \forall x \in [a, b]$$

Let $x_0 \in [a, b]$ be any point in [a, b]. Consider the two equations $c_1 y_1(x_0) + c_2 y_2(x_0) = y(x_0)$

and $\qquad c_1 y_1'(x_0) + c_2 y_2'(x_0) = y'(x_0)$

The above two simultaneous equations in c_1 and c_2 will have a unique solution if the determinant

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$$

But

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = y_1(x_0)\, y_2'(x_0) - y_1'(x_0)\, y_2(x_0)$$

$$= W(y_1, y_2)(x_0)$$

$$\neq 0 \text{ using theorem 2.4.4.}$$

Thus y (x) and $c_1 y_1(x) + c_2 y_2(x)$ both have same value at x_0 and the derivative of both of them also has same value at x_0. By uniqueness of solution we must have $y(x) = c_1 y_1(x) + c_2 y_2(x)$ for all $x \in [a, b]$.

Thus for every solution $y(x)$ we are able to obtain constants c_1 and c_2 such that $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$\Rightarrow c_1 y_1(x) + c_2 y_2(x)$ is the general solution of (*) which completes the proof.

Illustration 6:

Show that e^x and e^{-x} are linearly independent solutions of $y'' - y = 0$ on any interval.

Solution:

Let $y_1(x) = e^x$ and $y_2(x) = e^{-x}$

$y_1'(x) = e^x$, $y_1''(x) = e^x$

$y_1''(x) - y_1 = e^x - e^x = 0$ Hence $y_1(x) = e^x$ is a solution to $y'' - y = 0$

Further $y_2'(x) = -e^{-x}$ and $y_2''(x) = e^{-x}$

and $y_2''(x) - y_2(x) = e^{-x} - e^{-x} = 0$ Hence $y = (x) = e^{-x}$ is a solution to

$y'' - y = 0$

$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = e^x(-e^{-x}) - e^{-x}(e^x) = -1 - 1 = -2 \neq 0$

Hence y_1 and y_2 are linearly indpenedant solution of the equation

$y'' - y = 0$

Illustration 7:

Show that $y = c_1 x + c_2 x^2$ is the general solution of $x^2 y'' - 2xy' + 2y = 0$ on any interval not containing zero and find the particular solution for which $y(1) = 3$ and $y'(1) = 5$.

Solution:

Let $y_1(x) = x$ and $y_2(x) = x^2$

$y_1'(x) = 1$ $y_1''(x) = 0$

$x^2 x_1'' - 2xy_1' + 2y_1 = x^2(0) - 2x \times 1 + 2 \cdot x = 0$

$\therefore y_1(x)$ is a solution of the given differential equation

$y_2'(x) = 2x$ and $y_2''(x) = 2$

$x^2 y_2'' - 2x y_2' + 2y_2 = x^2(2) - 2x(2x) + 2x^2 = 0$

$\therefore y_2(x)$ is a solution of the given differential equation

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

$$= x(2x) - x^2(1) = x^2 \neq 0 \text{ provided } x \neq 0$$

Hence Wrongkian is not zero in any interval not containing zero. Hence y_1 and y_2 are linearly independent solutions. By theorem 2.5.1,

$y = c_1 x + c_2 x^2$ is the general solution of the given differential euqaiton.

$$y' = c_1 + 2x\, c_2$$

$$y'(1) = c_1 + 2\, c_2$$

Solving $y(1) = 3$ and $y'(1) = 5$

Solving simultaneously we get

$$c_1 + c_2 = 3$$

$$c_1 + 2c_2 = 5$$

$$c_2 = 2 \text{ and } c_1 = 1$$

$\therefore$ The particular solution is $y(x) = x + 2x^2$

2.5.2 Use of a known Solution to find another:

Suppose $y_1(x)$ is a known solution of the differential equation $y'' + P(x)y' + Q(x)y = 0$.

We will find another solution $y_2(x)$ to this equation so that $y_1(x)$ and $y_2(x)$ are linearly independent.

Theorem:

If $y_1(x)$ is a non zero solution of the equation

$$y'' + P(x)y' + Q(x)y = 0 \qquad \dots (*)$$

then $y_2(x) = y_1(x) \int \frac{1}{y_1^2} e^{\int P(x)\,dx}\, dx$ is another solution of (*)

so that $y_1(x)$ and $y_2(x)$ are linearly independent.

Proof:

Let y = v(x) y(x) be a solution of (*)

so that $y_2'' + Py_2' + Qy_2 = 0$ ... (*)

Substituting $y_2 = v\, y_1$

$y_2' = v\, y_1' + v'\, y_1$

and $y_2'' = v\, y_1'' + 2v'\, y_1' + v''\, y_1'$

Substituting in (1)

$vy_1'' + 2\, v'\, y_1' + v''y_1 + P\,(v\, y_1' + v'\, y_1) + Q\,(v\, y_1) = 0$

$v\,(y_1'' + Py_1' + Qy_1) + v''y_1 + v'\,(2y_1' + P\, y_1) = 0$

But y_1 is a solution of (*) $\Rightarrow y_1' + Py_1' + Q\, y_1 = 0$

$\Rightarrow v''\, y_1 + v'\,(2y_1' + Py_1) = 0$

$$\Rightarrow \frac{v''}{v'} = -\frac{(2y_1' + Py_1)}{y_1} = -\frac{2y_1'}{y_1} - P$$

Integrating $\log v' = -2 \log y_1 - \int P dx$

$$\log v' + 2 \log y_1 = -\int P dx.$$

$$\log v' + \log y_1^2 = \int P dx$$

$$\log v'\, y_1^2 = -\int P dx$$

$$v'\, y_1^2 = e^{-\int P(x)\, dx}$$

$$v' = \frac{1}{y_1^2} e^{-\int P\, dx}$$

$$v = \int \frac{1}{y_1^2} e^{-\int P\, dx}\, dx$$

$$y_2 = v(x)\, y_1(x) \text{ where } v(x) = \int \frac{1}{y_1^2} e^{-\int P(x)\,dx}\, dx$$

Now we show that $y_1(x)$ and $y_2(x)$ are linearly independent.

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_2 y_1' \\ &= y_1 (v y_1' + v' y_1) - v y_1 y_1' \\ &= v' y_1^2 \end{aligned}$$

But $v' = \frac{1}{y_1^2} e^{-\int P\,dx}$ is never zero being exponential function and $y_1^2 \neq 0$.

$\Rightarrow W(y_1, y_2)$ is never zero

$\Rightarrow y_1$ and y_2 are linearly independent.

Illustration 8:

Find the general solution of the following equation given $y_1(x) = \sin x$ is a solution of the differential equation $y'' + y = 0$.

Solution:

Let $y_2(x) = v(x) \sin x$ be a solution to the equation $y'' + y = 0$

Then $\quad v(x) = \int \frac{1}{y_1^2} e^{-\int p\,dx}\, dx$

Here $\quad p(x) = 0$ and $y_1(x) = \sin x$

$$v(x) = \int \frac{1}{\sin^2 x} e^0\, dx = \int \operatorname{cosec}^2 x\, dx$$

$$= -\cot x$$

$$y_2(x) = -\cot x \sin x = -\cos x$$

$$y(x) = c_1 \sin x + c_2(-\cos x) = c_1 \sin x + c_2' \cos x \text{ is}$$

the general solution of $y'' + y = 0$.

Illustration 9:

Find the other linearly independent solution to the differential equation $(1 - x^2)y'' - 2xy' + 2y = 0$ given $y_1 = x$ is a solution.

Solution:

$$y_2(x) = v(x)\, y_1(x)$$

where $$v(x) = \int \frac{1}{y_1^2} e^{-\int P\,dx}\, dx$$

Rewriting the equation in the standard form

$$y'' - \frac{2x}{1-x^2} y' + \frac{2}{1-x^2} y = 0 \quad \text{Here } p(x) = \frac{-2x}{1-x^2}$$

$$e^{-\int P\,dx} = e^{\int \frac{2x}{1-x^2} dx}$$

$$= e^{-\log(1-x^2)} = \frac{1}{1-x^2}$$

$$v(x) = \int \frac{1}{x^2} \cdot \frac{1}{1-x^2}\, dx$$

$$= \int \frac{1}{x^2(1-x^2)}\, dx$$

$$= \int \frac{1-x^2+x^2}{x^2(1-x^2)}\, dx$$

$$= \int \frac{1}{x^2} dx + \frac{1}{1-x^2} dx$$

$$= \frac{x^{-1}}{-1} + \log\left|\frac{1-x}{1+x}\right|$$

$$= -\frac{1}{x} + \log\left|\frac{1-x}{1+x}\right|$$

$$y_2(x) = x\, v(x) = -1 + x \log\left(\frac{1-x}{1+x}\right)$$

The general solution is $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$= c_1 x + c_2\left(x \log\left(\frac{1-x}{1+x}\right) - 1\right).$$

EXERCISE 2.2

(1) Show that $y = c_1 e^x + c_2 e^{2x}$ is the general solution of $y'' - 3y^1 + 2y = 0$ on any interval and find the particular solution for which $y(0) = -1$ and $y'(0) = 1$.

(2) Show that $y = c_1 e^{2x} + c_2 e^{2x}$ is the general solution of $y'' - 4y' + 4y = 0$ on any interval and find the particulars solution for which $y(0) = 2$ and $y'(0) = 1$.

(3) Verify that $y_1 = x^{-\frac{1}{2}}$ is a solution of the equation $x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$ over any interval of positive values of x and find the general solution of the differential equation.

(4) Prove that $y_1 = x$ is a solution of the following equations and find their genenral solution.

(i) $y'' - \frac{x}{x-1} y' + \frac{1}{x-1} y = 0$

(ii) $x^2 y'' + 2xy' - 2y = 0$

(5) Find the linearly independent solution of the equation

$xy'' - (2x + 1) y' + (x + 1) y = 0$ given that $y_1 = e^x$ is a solution.

(6) Show tht $y = c_1 (\sin x - \cos x) + c_2 e^{-x}$ is the general solution of

$y'' + (1 - \cos x) y' - y \cot x = 0$.

(7) Find the general solution of $\sin^2 x\, y'' - 2y = 0$ given that $y = \cot x$ is a solution.

(8) Find the general solution of $x^2 y'' + xy' - 9y = 0$ given that $y = x^3$ is a solution.

(9) Verify that $(1 - x^2)$ is a solution of the equation

$x(1 - x^2)^2 y'' - (1 - x^2)(1 - 3x^2) y' + 4x(1 + x^2) y = 0$

and find the general function.

ANSWER 2.2

(1) $y = -3e^x + 2e^{2x}$

(2) $y = 2e^{2x} - 3xe^{2x}$

(3) $y = c_1 x^{-\frac{1}{2}} \sin x + c_2 x^{-\frac{1}{2}} \cos x$

(4) (i) $y = c_1 x + c_2 e^x$ (ii) $y = c_1 x + c_2 x^{-2}$

(5) $y = c_1 e^x + c_2 x^2 e^x$

(7) $y = c_1 \cot x + c_2 (1 - x \cot x)$

(8) $y = c_1 x^3 + c_2 x^{-3}$

(9) $y = (1 - x^2) [c_1 + c_2 \log (1 - x^2)]$